Appendix F

Reliability Block Diagram (RDB)

Reliability block diagrams are valuable when we want to visualise the performance of a system comprised of several (binary) components.

Figures **??** and **??** shows the reliability block diagram for simple structures. The interpretation of the diagram is that the system is functioning if it is a connection between a and b, i.e., there exists a path of functioning components from a to b. The system is in a fault state (is not functioning) if it does not exist a path of functioning components between a and b.

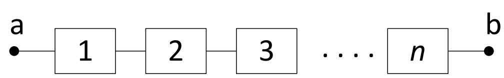

Figure F.1: Reliability block diagram for a series structure

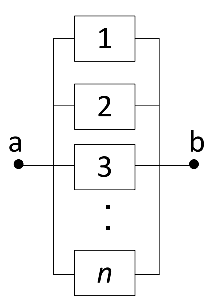

Figure F.2: Reliability block diagram for a parallel structure

F.1 Structure function

Recall that we for components have

$$x_i(t) = \begin{cases} 1 \text{ if component } i \text{ is functioning at time } t \\ 0 \text{ if component } i \text{ is in a fault state at time } t \end{cases} \tag{F.1}$$

For the system we now introduce

$$\phi(\mathbf{x}, t) = \begin{cases} 1 \text{ if the system is functioning at time } t \\ 0 \text{ if the system is in a fault state (not functioning) at time } t \end{cases} \tag{F.2}$$

ϕ denotes the structure function, and depends on the x_is ($\mathbf{x}$ is a vector of all the x_i's). $\phi(\mathbf{x}, t)$ is thus a mathematical function that uniquely determines whether the system functions or not for a given value of the $\mathbf{x}$ -vector. Note that it is not always straight forward to find a mathematical expression for $\phi(\mathbf{x}, t)$.

To simplify notation we skip the index t in the following.

F.2 The structure function for some simple structures

In the following we omit the time dependence from the notation.

For a series structure we have:

$$\phi(\mathbf{x}) = x_1 \cdot x_2 \cdot \ldots \cdot x_n = \prod_{i=1}^{n} x_i$$

For a parallel structure we have

$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)\ldots(1 - x_n) = 1 - \prod_{i=1}^{n}(1 - x_i) = \coprod_{i=1}^{n} x_i$$

Note that we for two components in parallel may simplify:

$$\phi(x_1, x_2) = x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2$$

where the notation $\amalg$ ("ip") is used for the co-product. For a k-out-of-n ($k$oo$n : G$) structure we have

$$\phi(\mathbf{x}) = \begin{cases} 1 \text{ if } \sum_{i=1}^{n} x_i \geq k \\ 0 \text{ if } \sum_{i=1}^{n} x_i < k \end{cases}$$

A k-out-of-n system is a system that functions if and only if at least k out of the n components in the system is functioning. We often write $k$oo$n$ *to* denote a k-out-of-n system, for example 2oo3.

The expression for the structure function of a k-out-of-n structure is not attractive from a calculation point of view, i.e., we cannot multiply this expression with other parts of the structure function. We may instead represent the structure by a parallel structure where each of the parallels comprises a series structure of k components. Such a parallel will then function if k or more components are functioning. There are altogether $\binom{n}{k}$ ways we may choose k components out of n components, hence we will have $\binom{n}{k}$ branches. As an example a 2-out-of-3 system may be written as $\binom{3}{2} = 3$ parallels, i.e., {1,2},{1,3} and {2,3}.

For structures comprised of series and parallel structures we may combine the above formulas by splitting the reliability block diagram into sub-blocks, and then apply the formulas within a block, and then treat a sub-block as a block on a higher level. Figure **??** shows how we may

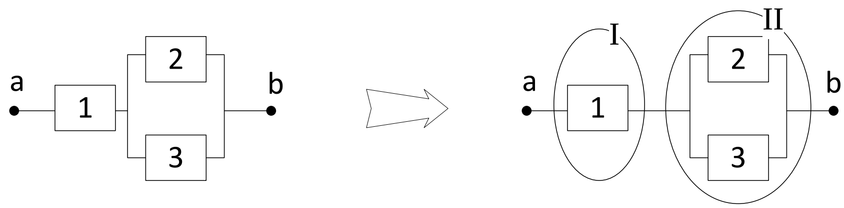

Figure F.3: Splitting the reliability block diagram in sub-blocks

split the reliability block diagram into sub-blocks, here I and II. We may then write $\phi(x) = \phi_{\text{I}} \times \phi_{\text{II}}$ because I and II are in series. Further, we have $\phi_{\text{I}} = x_1$, and $\phi_{\text{II}} = 1\text{-}(1\text{-}x_2)(1\text{-}\ x_3)$, thus we have $\phi(\mathbf{x}) = x_1(1\text{-}(1\text{-}x_2)(1\text{-}\ x_3))$.

F.3 System structure analysis

Coherent structures

A system of components is said to be coherent if all its components are relevant and the structure function is non-decreasing in each argument. Proofs are given in the textbook.

The results that follow are only valid for coherent structures.

F.3.1 Basic results for coherent structures

Property 4.1

$$\phi(\mathbf{0}) = 0 \text{ and } \phi(\mathbf{1}) = 1$$

Property 4.2 - Any structure is "between" the series and parallel:

$$\prod_{i=1}^{n} x_i \le \phi(\mathbf{x}) \le \coprod_{i=1}^{n} x_i$$

Property 4.3 The effect of redundancy is higher on component level than on system level

$$\phi(\mathbf{x} \amalg \mathbf{y}) \geq \phi(\mathbf{x}) \amalg \phi\mathbf{y})$$

We also have:

$$\phi(\mathbf{x} \cdot \mathbf{y}) \leq \phi(\mathbf{x}) \cdot \phi(\mathbf{y})$$

Paths and Cuts

As for fault tree analysis, we may define cut- and path sets for a structure of n components: $C = \{1, 2, \ldots, n\}$

- A cut set K is a set of components in C which by failing causes the system to fail. A cut set is minimal if it cannot be reduced without loosing its status as a cut set.
- A paht set P is a set of components in C which by functioning ensures that the system is functioning. A path set is minimal if it cannot be reduced without loosing its status as a path set.

F.3.2 Structure Represented by Minimal Path Series Structures

A path, say P_j, may be considered as a series structure. Since any functioning path ensures the system to function, each path may be considered as one branch in a parallel structure of all the minimal paths, hence we have:

$$\phi(\mathbf{x}) = \coprod_{j=1}^{p} \prod_{i \in P_j} x_i$$

F.3.3 Structure Represented by Minimal Cut Parallel Structures

A cut, say K_j, may be considered as a parallel structure. Since any failed cut ensures the system to fail, each cut may be considered as one element in a series structure of all the minimal cuts, hence we have:

$$\phi(\mathbf{x}) = \prod_{j=1}^{k} \coprod_{i \in K_j} x_i$$

Pivotal Decomposition

Pivotal decomposition is often used to analyse complex structures where one or more components are causing "trouble" when we search for series and parallel structures. Typically we use this method for solving “bridge structures”. Introduce the following notation:

- $\phi(1_i, \mathbf{x})$ = The structure function of the structure when it is given that component i is in a functioning state, i.e., $x_i = 1$.
- $\phi(0_i, \mathbf{x})$ = The structure function of the structure when it is given that component i is in a fault state, i.e., $x_i = 0$.

We then have:

$$\phi(\mathbf{x}) \equiv x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x}) \text{ for all} \mathbf{x}$$

This result is often used when obtaining the structure function of a complex structure. The idea is to use pivotal decomposition of the component that makes the structure troublesome. Conditioning on that component is functioning, i.e., $x_i = 1$, we may rather easily obtain the structure function of the remaining structure, i.e., $\phi(1_i, \mathbf{x})$, and similarly if $x_i = 0$, we may rather easily obtain the structure function of the remaining structure, i.e., $\phi(0_i, \mathbf{x})$, and then the result for pivotal decomposition may be applied.

F.4 Summary: Finding the structure function

The following principles may be used to find the structure function for a reliability block diagram:

- For a series structure we have $\phi(\mathbf{x}) = x_1 \cdot x_2 \cdot \ldots \cdot x_n = \prod_{i=1}^{n} x_i$.
- For a parallel structure we have $\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2)\ldots(1 - x_n) = 1 - \prod_{i=1}^{n}(1 - x_i) = \coprod_{i=1}^{n} x_i$.
- For a k-out-of-n structure we may represent the structure by a parallel structure where each of the parallels comprises a series structure of k components. There are altogether $\binom{n}{k}$ ways we may choose k components out of n components, hence we will have $\binom{n}{k}$ branches.
- For bridge structures and other structures where it is not easy to “see” series and parallel structures, we may use pivotal decomposition, i.e., $\phi(\mathbf{x}) \equiv x_i \phi(1_i, \mathbf{x}) + (1 - x_i)\phi(0_i, \mathbf{x})$ where we decompose around the “troublesome” component.
- If the minimal cut sets are available for a system (or a sub system), the corresponding structure function is given by: $\phi(\mathbf{x}) = \prod_{j=1}^{k} \coprod_{i \in K_j} x_i$, and similarly for the path sets: $\phi(\mathbf{x}) = \coprod_{j=1}^{p} \prod_{i \in P_j} x_i$.

- We may identify sub-blocks in the diagram (modules), where we for each module represent the state variable of the module by a structure function. i.e., a function of the elements in the module, which is also binary, hence a "formula" can replace the module as if it was a component. In principle any combination of series and parallel structure may be analysed to get the structure function. Bridge structures and k-out-of-n structures may also be handled this way. The same principle may also be used if a sub-block is represented by its minimal cut sets or minimal path sets.

F.5 Quantitative calculations

Up to now we have mainly used the symbol x_i to represent the value a state variable may take. In order to assess component and system reliability, we need to treat the state variables and the structure function as random quantities (stochastic variables). We let $X_i(t)$ denote the state variable i, and $\mathbf{X}(t) = (X_1(t), X_2(t), ..., X_n(t))$ be the state vector. Further the structure function is now a random quantity, i.e., $\phi(\mathbf{X}(t))$. Now, introduce the following probabilities:

$$p_i(t) = \Pr(X_i(t) = 1) = \text{ Component reliability}$$
$$p_S(t) = \Pr(\phi(\mathbf{X}(t)) = 1) = \text{ System reliability}$$

Since both the state variables and the structure function is binary we have:

$$\mathrm{E}\left[X_i(t)\right] = p_i(t)$$
$$\mathrm{E}\left[\phi(\mathbf{X}(t))\right] = p_S(t)$$

Since the system reliability $p_S(t)$ depends on the component reliabilities, we often write:

$$p_S(t) = h[p_1(t), p_2(t), \ldots, p_n(t)] = h[\mathbf{p}(t)]$$

F.5.1 Reliability of series structures

Since the structure function of a series structure is the product of the state variables, we have

$$h[\mathbf{p}(t)] = \mathrm{E}\left[\phi(\mathbf{X}(t))\right] = \mathrm{E}\left[\prod_{i=1}^{n} X_i(t)\right] = \prod_{i=1}^{n} \mathrm{E}\left[X_i(t)\right] = \prod_{i=1}^{n} p_i(t)$$

where we have used that the expected value of a product equals the product of the expectations *if* the stochastic variables are *independent*.

F.5.2 Reliability of parallel structures

A similar argument may be used for a parallel structure which gives:

$$h[\mathbf{p}(t)] = 1 - \prod_{i=1}^{n} \left[1 - p_i(t)\right] = \coprod_{i=1}^{n} p_i(t)$$

In a more general setting assume that we are able to write the structure function as a *sum of products* of the state variables. Further assume *independent* components and that we have removed any exponents in the expressions, i.e., $x_i^n = x_i$. We then use the results that "the expectation of a sum equals to the sum of expectations" and "the expectation of a product equals the product of the expectations". This means that $p_S(t) = \mathrm{E}\left[\phi(\mathbf{X}(t))\right]$ will equal the sum of products of expectations, i.e., a sum of products of $\mathrm{E}[X_i(t)]$'s. Further since $\mathrm{E}[X_i(t)] = p_i(t)$ we have proven that the system reliability $p_S(t)$ may be found by replacing all the x_i''s in the structure function with corresponding $p_i(t)$'s.

Note that this approach is only valid if we have carried out the multiplication, i.e., resolved any parentheses in the expression for the structure function, and removed any exponents.

F.5.3 General approach utilizing the structure function

1. Map the physical system into a reliability block diagram or another representation as a starting point

2. Use various approaches (series, parallels, bridges, k-out-of-n's etc) to derive the structure function

3. Multiply out any parentheses, collect terms, and remove any exponents, yielding a structure function as a sum of products

4. The system reliability, $p_S(t)$ is now found by replacing all the x_i's with corresponding $p_i(t)$'s in the sum of product version of the structure function

Note that exponents in the structure function can always be removed because $x^n = x$ for all binary variables. Further exponents shall always be removed since our expectation results above is not valid if there are exponents corresponding to dependent variables, for example $X^2 = X \cdot X$ is the product of the same variable.

Example

Assume that we have component 1 in series with a parallel of two components 2 and 3 as shown in Figure **??**

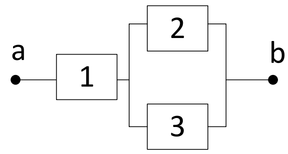

Figure F.4: Example RBD

The structure function is

$$\phi(\mathbf{x}) = x_1 \cdot (x_2 \amalg x_3) = x_1 x_2 + x_1 x_3 - x_1 x_2 x_3$$

We now replace all the x_i''s in the structure function with corresponding $p_i(t)$'s to get the system reliability. Assuming $p_1 = 0.99$, $p_2 = p_3 = 0.9$ gives:

$$p_S(t) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3 = 0.891 + 0.891 - 0.8019 = 0.9801$$