TPK4120 - Lecture summary

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Chapter 11 - Markov Analysis

What to know

- The definition of a Markov process, and what is meant by homogeneous transition probabilities
- That it is possible to derive the Markov differential equations, but we do not need to know all the details
- The understanding of states and how we derive the Markov transition diagram
- How to map the information in the Markov transition diagram into the transition matrix, $\mathbf{A}$
- The understanding of the Markov differential equations: $\mathbf{P}(t) \cdot \mathbf{A} = \dot{\mathbf{P}}(t)$
- How to find the steady state solution: (i) Steps required for the numerical solution, (ii) that it is possible to find an analytical solution....
- The definition of the visiting frequencies, and how to find them
- That the time dependent solution is given by $\mathbf{P}(t) = \mathbf{P}(0)e^{t\mathbf{A}}$ which cannot be used unless a comprehensive matrix library is available
- How to use the iterative scheme: $\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)[\mathbf{A}\Delta t + \mathbf{I}]$ if we have access to a computer with simple matrix functions

Definitions

A Markov process is a special type of stochastic processes where the process posses the so-called Markov property. A stochastic process $\{X(t), t \in \Theta\}$ is a collection of random variables. The set Θ is called the *index set* of the process. For each index t in Θ, $X(t)$ is called the *state* of the process at time t.

In the general presentation we always assume that $X(t)$ can only take the values $1, 2, ..., r$. In practical examples it is often convenient to allow for an additional zero value for the state variable. A process is said to have the Markov property if:

$$\Pr(X(t+s) = j | X(s) = i \cap \text{ some history up to time } s) = \Pr(X(t+s) = j | X(s) = i)$$

This means that given the process is in state i at some time s, the probability of being in another state, say j, t time units later is independent of the history up to time s, i.e., we may ignore all information regarding the process in the past when looking into the future. The only thing that counts is the current state.

This general presentation also only treats Markov processes with *stationary transition probabilities*. This means that:

$$\Pr(X(t+s) = j | X(s) = i) = \Pr(X(t) = j | X(0) = i) \text{ for all } s, t \geq 0$$

that is, the probability of going from state i to j during a time period of t is independent of the starting point of such a "journey".

The following notation is introduced:

$$P_{ij}(t) = \Pr(X(t) = j | X(0) = i)$$

The so-called sojourn time, $\tilde{T}_i$, is the time the process spends in state i from it arrives to state i before it jumps out of state i. Further let T_{ij} denote the time the process spends in state i before it eventually jumps to state j. The *transition rate* from state i to state j is denoted a_{ij} and is the limiting conditional probability of jumping to state j given that the process is in state i (divided by the length of the interval considered). It may be argued that the Markov property and the stationary transition probabilities yields that all transition times are *exponentially distributed*. The total rate of transition out of state i is denoted α_i, where

$$\alpha_i = \sum_{j \neq i} a_{ij}$$

From the fact that the sojourn time and all other transition times are exponentially distributed it follows that:

$$P_{ii}(\Delta t) = \Pr(\tilde{T}_i > \Delta t) = e^{-\alpha_i \Delta t} \approx 1 - \alpha_i \Delta t$$
$$P_{ij}(\Delta t) = \Pr(T_{ij} \leq \Delta t)) = 1 - e^{-a_{ij} \Delta t} \approx a_{ij} \Delta t$$

Rearranging and letting Δt approach 0, we get:

$$\lim_{\Delta t \to 0} \frac{1 - P_{ii}(\Delta t)}{\Delta t} = \alpha_i \tag{1}$$

$$\lim_{\Delta t \to 0} \frac{P_{ij}(\Delta t)}{\Delta t} = a_{ij} \tag{2}$$

These two equations will later be used to obtain the Kolmogorov differential equations. From the Markov property and the law of total probability we have:

$$P_{ij}(t+s) = \sum_{k=1}^{r} P_{ik}(t)P_{kj}(s)$$

This equation is denoted the Chapman-Kolmogorov equations. We utilize this equation to find:

$$P_{ij}(t+\Delta t) = P_{ij}(\Delta t + t) = \sum_{k=1}^{r} P_{ik}(\Delta t)P_{kj}(t)$$

Rearranging (having in mind we are seeking the derivative) we get:

$$P_{ij}(t+\Delta t) - P_{ij}(t) = \sum_{\substack{k=1 \\ k\neq i}}^{r} P_{ik}(\Delta t)P_{kj}(t) - [1 - P_{ii}(\Delta t)]P_{ij}(t)$$

Now dividing by Δt, inserting equations 1 and 2, letting $\Delta t \to 0$, and defining $a_{ii} = -\alpha_i$, we get after some rearrangements:

$$\dot{P}_{ij}(t) = \sum_{k=1}^{r} a_{ik}P_{kj}(t) \tag{3}$$

These differential equations are denoted the *Kolmogorov backward equations*. Similarly, we may obtain the *Kolmogorov forward equations*:

$$\dot{P}_{ij}(t) = \sum_{k=1}^{r} a_{kj}P_{ik}(t) \tag{4}$$

The term 'backward' refers to that the equations were derived by considering an instant jump (transition) to state k back at the start of the interval, and then go to the required state j, i.e., first Δt and then t. The 'forward' equations are derived by first considering going from i to k during time t and then make an instant jump to the required state j at the end of the interval, i.e., first t and then Δt.

Markov state equations

We now assume that we know the initial state, and assume that the process started in state i. We then simplify notation by omitting the index for the initial state, hence we write $P_j(t)$ instead of $P_{ij}(t)$.

It is convenient to introduce matrix and vector notation. First we define

the transition rate matrix[1] , **A**:

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix}$$

where

$$a_{ii} = -\alpha_i = -\sum_{\substack{j=1 \\ j \neq i}}^{r} a_{ij}$$

which means that the diagonal elements are defined such that the sum of each row equals zero.

Further we define the row vectors: $\mathbf{P}(t) = [P_1(t), P_2(t), \ldots, P_r(t)]$ and $\dot{\mathbf{P}}(t) = [\dot{P}_1(t), \dot{P}_2(t), \ldots, \dot{P}_r(t)]$. We may then write the Kolmogorov forward equations on matrix format:

$$[P_1(t), P_2(t), \ldots, P_r(t)] \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix} = [\dot{P}_1(t), \dot{P}_2(t), \ldots, \dot{P}_r(t)]$$

that is:

$$\mathbf{P}(t) \cdot \mathbf{A} = \dot{\mathbf{P}}(t) \tag{5}$$

Time dependent solution for the Markov process

To solve Equation (5) as a function of time we may use an analogy to ordinary differential equations in one dimension and we get:

$$\mathbf{P}(t) = \mathbf{P}(0) e^{t\mathbf{A}}$$

Although this is a very elegant solution, it is not very attractive since taking the exponential of a matrix is not that easy. Computer codes such as Matlab is required. We may, however, rewrite Equation (5) as:

$$\dot{\mathbf{P}}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{P}(t + \Delta t) - \mathbf{P}(t)}{\Delta t} = \mathbf{P}(t) \cdot \mathbf{A}$$

yielding

$$\mathbf{P}(t + \Delta t) \approx \mathbf{P}(t)[\mathbf{A}\Delta t + \mathbf{I}] \tag{6}$$

[1] In this presentation boldface notation is used both for vectors and matrices, e.g., **P** and **A**. In the textbook boldface notation is used for vectors but Blackboard bold font, i.e., $\mathbb{A}$ are used for matrices

where $\mathbf{I}$ is the identity matrix. This equation may now be used iteratively with a sufficient small time interval Δt and starting point $\mathbf{P}(0)$ to find the time dependent solution. Only simple matrix multiplication is required. Implementing a solution in for example VBA some considerations are required regarding the step length Δt. Choosing a too low value gives numerical problems and will also require longer computational time. Choosing a too high step length will cause the approximation in Equation (6) to be inaccurate. A rule of thumb will be to use a value of one tenth of the inverse value of the highest transition rate.

Note that in Markov analysis we usually only require the time-dependent solution for a limiting time period, and typically we would like to calculate $\mathbf{P}(t)$ at values $t = 0, \Delta t, 2\Delta t, \ldots$. Using Equation (6) is therefore attractive. To improve the approximation in Equation (6) we could use one "intermediate" point, i.e., we could use:

$$\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)[\mathbf{A}\Delta t/2 + \mathbf{I}][\mathbf{A}\Delta t/2 + \mathbf{I}] \tag{7}$$

and even improve by splitting into 2^n sub-intervals, yielding:

$$\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)\left[\mathbf{A}\Delta t/2^n + \mathbf{I}\right]^{2^n} \tag{8}$$

Note the similarity between Equation (8) and Equation (11.106) in the textbook. The advantage of Equation (8) is the calculation efficiency, i.e., we only need n matrix multiplications to reduce the step-length by a factor 2^n. Note that we only calculate $[\mathbf{A}\Delta t/2^n + \mathbf{I}]^{2^n}$ once in Equation (8), so we could afford double precision in that part of the calculations to increase the precision. It should be noted that there is still a trade-off between round-off errors and accuracy in the approximation in Equation (8), and a good choice of n would be in the range 4-6.

Steady state solution for the Markov process

In the long run we will have that $\dot{\mathbf{P}}(t) \to \mathbf{0}$ when $t \to \infty$, hence $\mathbf{P}(t) \cdot \mathbf{A} = \mathbf{0}$. We define the steady state probabilities by the vector $\mathbf{P} = [P_1, P_2, \ldots, P_r]$, where we have omitted the time dependency (t) to reflect that in the long run the state probabilities are not changing any more.

To solve the steady state equations we realize that the matrix $\mathbf{A}$ has not full rank due to the way have have established the diagonal elements. To overcome this problem we remove one (arbitrary) of the equations in the following set of equations:

$$[P_1, P_2, \ldots, P_r] \cdot \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1r} \\ a_{21} & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ a_{r1} & a_{r2} & \cdots & a_{rr} \end{bmatrix} = [0, 0, \ldots, 0]$$

and replace it by the following equation:

$$\sum_{j=1}^{r} P_j = 1$$

For example replacing the first equation gives:

$$[P_1, P_2, \ldots, P_r] \cdot \begin{bmatrix} 1 & a_{12} & \cdots & a_{1r} \\ 1 & a_{22} & \cdots & a_{2r} \\ \vdots & \vdots & \cdots & \vdots \\ 1 & a_{r2} & \cdots & a_{rr} \end{bmatrix} = [1, 0, \ldots, 0]$$

In matrix form we write:

$$\mathbf{P} \cdot \mathbf{A}_1 = \mathbf{b} \tag{9}$$

where $\mathbf{b}$ is a row vector of zeros except for the first element which equals one. Note that Equation (9) is not on standard form $\mathbf{A} \cdot \mathbf{x} = \mathbf{b}$. Transposing each side on the equal symbol in Equation (9) gives $\mathbf{A}_1^{\mathrm{T}} \cdot \mathbf{P}^{\mathrm{T}} = \mathbf{b}^{\mathrm{T}}$ which could be solved by standard Gauss-Jordan elimination.

Ideally we could obtain an analytical solution for the steady state equations, but for $r > 3$ we usually stick to numerical solutions.

Visit frequency

The visiting frequency, ν, is one of several system performance that we define for the steady-state situation. The visiting frequency for state j, ν_j, is the unconditional transition rate into state j. We could make different arguments for the arrival rate, say ${\nu_j}^{\mathrm{arr}}$, and the departure rate, say ${\nu_j}^{\mathrm{dep}}$. Considering departures we may argue directly that:

$${\nu_j}^{\mathrm{dep}} = \alpha_j P_j \tag{10}$$

Similarly for arrival we have from the law of total probability:

$${\nu_j}^{\mathrm{arr}} = \sum_{k \neq j} P_k a_{kj} \tag{11}$$

Since in the long run we should fulfil the *balance equations* stating that the total rate into a state equals the total rate out of that state we get:

$$\nu_j = \alpha_j P_j = \sum_{k \neq j} P_k a_{kj}$$

Mean time to first passage to a given state

The visiting frequency ν_j is the unconditional transition rate into state j, whereas $1/\nu_j$ is the unconditional mean time between state j is visited. In some situation we would rather find the mean time to the first time the system enters state j. To solve this problem we can make state j an *absorbing state*. An absorbing state means that we can not leave that state. To make a state absorbing we just remove all arcs out of that state.

Since we are considering state j as an absorbing state, we obtain the transition rate matrix identical with the original transition state matrix, except that the j'th row (corresponding to a departure) comprises only zeros. From before we know that the transition matrix has not full rank, and we may therefore remove any of the equations. It is convenient to remove the j'th column of the matrix. Further, since row j only contains zeros, $P_j(t)$ will disappear from all equations. We may therefore also remove the j'th row in the transition rate matrix. The result is a set of $r-1$ differential equations with $r-1$ unknowns, $P_1(t),\ldots,P_{j-1}(t),P_{j+1}(t),\ldots,P_r(t)$.

Note that when establishing the reduced system by removing the j'th row and the j'th column, the underlying argument is that we treat the modified system with j as an *absorbing state*, we can not do this in general. The reduced matrix is denoted $\mathbf{A}_{\mathrm{R}}$.

To solve the set of differential equations we introduce the Laplace transform. The Laplace transform of a function, say $f(t)$ is given by $f^*(s) = \mathscr{L}f(t) = \int_0^\infty e^{-st}f(t)dt$. The following rule applies for the Laplace transform:

$$\mathscr{L}[f'(t)] = s\mathscr{L}[f(t)] - f(0) = sf^*(s) - f(0)$$

In addition we have that the Laplace transform of a sum of functions equals the sum of the Laplace transforms of those functions. Now taking the Laplace transform on both sides of the set of differential equations, we observe that the right hand side is the derivative of the state probabilities, hence the Laplace of the right hand side will be $sP_i^*(s) - P_i(0)$, where $P_i(0) = 1$ only for the initial state, and 0 else.

The result is a set of $r-1$ linear equations with $r-1$ unknowns, $P_1^*(s),\ldots,P_{j-1}^*(s)$, $P_{j+1}^*(s),\ldots,P_r^*(s)$. In principle we may solve these equations by elimination, or we just use the solver in our linear algebra library.

The Laplace transform of the survivor function is

$$R^*(s) = \sum_{i=1,i\neq j}^{r} P_i^*(s)$$

If we are able to take the inverse Laplace transform, we may also find the survivor function $R(t)$ of the system. A trick to do this would be to arrange the denominator on the form $(s-k_1)(s-k_2)$ and then factorize, and hope that we get something we recognize from the table of Laplace transforms of known functions.

Our objective, is however, to find the mean time to the first time the system enters state j. We have that

$$\mathrm{E}(T) = \int_0^\infty R(t)dt$$

Further we also have

$$R^*(s) = \mathscr{L}R(t) = \int_0^\infty e^{-st}R(t)dt$$

Thus, by inserting $s = 0$ we have $\mathrm{E}(T) = \int_0^\infty R(t)e^0 dt = R^*(0)$.

Since $R^*(0) = \sum_{i=1, i\neq j}^{r} P_i^*(0)$ we therefore obtain the mean time to first system failure by:

$$\mathrm{MTTF} = \sum_{i=1, i\neq j}^{r} P_i^*(0)$$

Note that we by this procedure may establish the mean time to the first visit to sate j without actually calculating the Laplace transforms. What we actually do is to solve a set of linear equations, where the unknown variables are the $P_i^*(0)$'s from the reduced systems by removing the row column corresponding to the absorbing state. Further note that the right hand side equals 0 for all equations except the equation representing the initial state, where the right hand side equals -1, since $sP_i^*(s) = 0$ for $s = 0$.

Note that we here have assumed that state 0 represent the system failure. In a more general setting we apply the same approach but rather than deleting the first row and column to obtain the reduced matrix, we delete the rows and columns corresponding to one or more system failure states.

Birth-death processes

A birth-death process is a special type of Markov process where the transitions are to the next state immediately above or immediately below the current state. The states has some natural ordering, for example the number of customers being served by one or more servers. For that reason we also usually start the numbering from zero rather than one. The transition matrix is then tridiagonal as shown in Equation

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & 0 & \dots & \dots & \dots \\ a_{10} & a_{11} & a_{12} & 0 & \dots & \dots \\ 0 & a_{21} & a_{22} & a_{23} & 0 & \dots \\ \vdots & 0 & a_{32} & a_{33} & a_{34} & 0 \end{bmatrix} \tag{12}$$

The above-diagonal elements, $a_{ij}, j - i = 1$ are denoted births and causes the system state to increase by one, whereas the below-diagonal elements, $a_{ij}, i - j = 1$ are denoted deaths, and causes the system state to decrease by one. In birth-death processes it is common to use λ as a transition symbol for births and μ as transition symbol for deaths. A birth-death process may have a finite or an infinite number of states.

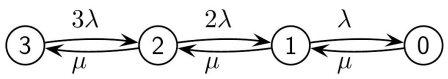

Figure 1: Markov transition diagram

Example

Consider a workshop with three critical machines. Each machine has a constant failure rate equal to λ and there is one repair man that can repair failed machines. The rate of repair is μ meaning that the mean repair time is $1/\mu$. The state variable represent the number of failed machines. The transition matrix is given by:

$$\mathbf{A} = \begin{bmatrix} ? & \mu & 0 & 0 \\ \lambda & ? & \mu & 0 \\ 0 & 2\lambda & ? & \mu \\ 0 & 0 & 3\lambda & ? \end{bmatrix}$$

Figure 1 shows the Markov transition diagram corresponding to the transition matrix.

Note when the system is in state 3 and all machines are functioning, there are 3 machines that potentially may fail, hence the transition rate from state 3 to state 2 equals 3λ. In state 2 there is only two machines that may fail, hence the transition rate from state 2 to state 1 is 2λ. Since there is only one repair man, all the above-diagonal elements equal the repair rate μ.

The question marks in the transition matrix represent the diagonal elements. They are completed at the end when all the “real" transitions are specified by applying the rule that all rows should sum to one, i.e., we get:

$$\mathbf{A} = \begin{bmatrix} -\mu & \mu & 0 & 0 \\ \lambda & -\lambda-\mu & \mu & 0 \\ 0 & 2\lambda & -2\lambda-\mu & \mu \\ 0 & 0 & 3\lambda & -\lambda \end{bmatrix}$$

Figure 2 shows the specification of this model in MS-Excel. It is convenient to give names to the cell containing λ and μ. The numerical values used are: $\lambda = 0.001$ and $\mu = 0.1$.

		To -->			
		0	1	2	3
From ↓	0		=mu	0	0
	1	=lambda		=mu	0
	2	0	=2*lambda		=mu
	3	0	0	=3*lambda	

Figure 2: MS Excel specification of the transition matrix

Table 1 shows the calculated steady state probabilities. Full production is achieved in 97% of the operating hours. For some 3% one machine is down for corrective maintenance, whereas the probability of two or more failed machines is very low.

Table 1: Steady state probabilities

State	P_i
3	0.9703
2	2.91E-02
1	5.82E-04
0	5.82E-06

Problem

In a workshop there are two production lines in parallel. Each production line has a critical machine with constant failure rate $\lambda = 0.01$ failures per hour. There is one (common) spare machine that can replace a failed machine. We assume that switching time can be ignored. The repair rate of the machines is assumed constant and equal to $\mu = 0.2$ per hour. If a production line is down the loss is assumed to be $c_U = 10\ 000$ NOKs per hour. Only one repair man is available.

- Construct the Markov diagram and find the steady state solution.
- Calculate the expected loss due to downtime.
- If production is not 24/7 but runs from 07:00 to 15:00 it is reasonable to assume that each morning we start with 3 functioning machines. Find the time dependent solution and find the expected loss due to downtime.
- Repeat the analysis, but assume that two repair men are available.
- How much should one be willing to pay per hour for having this extra backup on repair resources?

Procedure

The Markov Analysis is usually carried out in six steps:

1. Make a sketch of the system
2. Define the system states
3. Group similar sates to one state (reduce dimension)
4. Draw the Markov diagram with the transition rates

5. Quantitative assessment

6. Compilation and presentation of the result from the analysis

Below we describe the state together with an example

Step 1 - Make a sketch of the system

The sketch is mainly used to visualise parallel and serial structures, stand-by systems, switching systems etc. In Figure 3 we have drawn a sketch of a simple cold standby system. We consider a system comprising an active pump and a spare pump in cold stand-by. If the active pump fails, the stand-by pump is started and continue to do the duty. The failed active pump is then repaired. If the stand-by pump, which now is working, fails during the repair of the failed pump there will be a system failure.

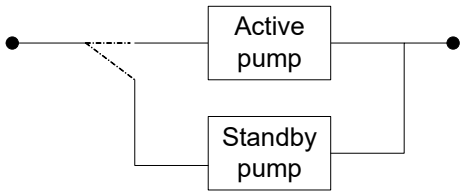

Figure 3: Example of cold standby system

Step 2 - Define the system states

Based on the sketch of the systems the various components are identified. For each component one or more states are defined. Often a number is given to each state, where the highest number represents perfect performance, whereas zero represent a complete fault state. Next the various states of all components are combined. This may lead to very many states due to the combinatorial effect. Table 2 shows the states for the example system.

Table 2: States for the example system

State	Explanation
2	Active pump is running
1	Active pump failed, stand-by pump running
0	Both pumps failed

Step 3 - Group similar sates to one state (reduce dimension)

This step is only introduced in order to reduce the dimension of the problem. In many situations several components may be identical and it will usually

be possible to group similar system states into one system state, and hence reduce the dimension of the problem. For example if we have $n = 3$ pumps we can group into state 3={All 3 pumps are OK}, 2={2 pumps are OK}, 1={One pump is OK} and 0={All pumps are failed}. For state 1 and 2 we do not distinguish between which pumps are functioning. In the example there is no need to group states.

Step 4 - Draw the Markov diagram with the transition rates

The various system states are now drawn in a Markov diagram. Each state is drawn as a circle labelled with the state number. Transitions between the states are visualised by drawing arrows between the corresponding circles. On each arrow the transition rate is labelled. Very often the Greek letter λ represents component failure rates, whereas the Greek letter μ represents repair rates.

Table 3 shows the transition rates for the example system. Here we assume that both pumps are repaired as part of one common repair activity if we enter state 0. Figure 4 shows the transition diagram for the example system.

Table 3: Transition rates

Rate	Explanation
λ_1	failure rate of the active pump
λ_2	failure rate of the standby pump (while running, $\lambda_2 = 0$ in standby position)
μ_1	repair rate of the active pump
μ_B	repair rate when both pumps are in a fault state

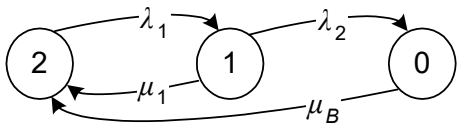

Figure 4: Transition diagram for the example system

Step 5 - Quantitative assessment

Steady state solution:
The transition matrix **A** is given by:

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\mu_B & 0 & \mu_B \\ \lambda_2 & -\lambda_2 - \mu_1 & \mu_1 \\ 0 & \lambda_1 & -\lambda_1 \end{bmatrix}$$

To find the steady state solution we solve the system $\mathbf{P}\cdot\mathbf{A}_1 = \mathbf{b}$ where we may replace any column (equation) in $\mathbf{A}$ with ones. To simplify the set of equations as much as possible we now choose to replace the last column:

$$[P_0, P_1, P_2]\cdot\begin{bmatrix} -\mu_B & 0 & 1 \\ \lambda_2 & -\lambda_2-\mu_1 & 1 \\ 0 & \lambda_1 & 1 \end{bmatrix} = [0,0,1]$$

To solve the set of equation we start with the second equation:

$$P_1(-\lambda_2-\mu_1)+P_2\lambda_1 = 0 \Rightarrow P_1 = \lambda_1/(\lambda_2+\mu_1)P_2$$

Inserted in the first equation:

$$P_0(-\mu_B)+P_1\lambda_2 = 0 \Rightarrow P_0 = \lambda_2/\mu_B P_1 = \frac{\lambda_1\lambda_2}{\mu_B(\lambda_2+\mu_1)}P_2$$

Now P_0 and P_1 may be inserted in the third equation:

$$P_0+P_1+P_2 = \left[\frac{\lambda_1\lambda_2}{\mu_B(\lambda_2+\mu_1)}+\frac{\lambda_1}{\lambda_2+\mu_1}+1\right]P_2 = 1$$

Multiplying with $\mu_B(\lambda_2+\mu_1)$ on both sides and rearranging gives:

$$P_2 = \frac{\mu_B(\lambda_2+\mu_1)}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$
$$P_1 = \frac{\mu_B\lambda_1}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$
$$P_0 = \frac{\lambda_1\lambda_2}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

Visiting frequencies:
From $\nu_j = -P_j a_j j$ we get for example:

$$\nu_0 = -P_0 a_{00} = -P_0(-\mu_B) = \frac{\mu_B\lambda_1\lambda_2}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

Time dependent solution:
The time dependent solution requires to solve the Laplace equations, and is rather complicated. Therefore we stick to numerical methods. At the end of this document we demonstrate the use of Laplace to find the time dependent solution for a simpler situation with only one component.

Mean time to first system failure:
We use the Laplace transform approach. That is, first we delete the row and

column corresponding to the absorbing state, i.e., state 0, and replace the P_j's with P_j^*s:

$$[P_1^*, P_2^*] \cdot \begin{bmatrix} -\lambda_2 - \mu_1 & \mu_1 \\ \lambda_1 & -\lambda_1 \end{bmatrix} = [0, -1]$$

with the solution $P_1^* = 1/\lambda_2$ and $P_2^* = (\lambda_2 + \mu_1)/(\lambda_1 \lambda_2)$, and thus:

$$\text{MTTF}_S = P_1^* + P_2^* = (\lambda_1 + \lambda_2 + \mu_1)/(\lambda_1 \lambda_2)$$

Time dependent solution for a repairable component

Consider a component with constant failure rate λ and constant repair rate μ. Let state 1 represent the functioning state and state 0 represent the failed state. The transition matrix for this system is given by:

$$\mathbf{A} = \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix}$$

Assuming the system starts in state 1 we have $P_0(0) = 0$ and $P_1(0) = 1$, and the Laplace transform of the time dependent solution is given by:

$$[P_0^*(s), P_1^*(s)] \begin{bmatrix} -\mu & \mu \\ \lambda & -\lambda \end{bmatrix} = [sP_0^*(s), sP_1^*(s) - 1]$$

Thus

$$\begin{aligned} -\mu P_0^*(s) + \lambda P_1^*(s) &= sP_0^*(s) \\ \mu P_0^*(s) - \lambda P_1^*(s) &= sP_1^*(s) - 1 \end{aligned}$$

Adding these two equations yields:

$$sP_0^*(s) + sP_1^*(s) = 1 \Rightarrow P_0^*(s) = 1/s - P_1^*(s)$$

and inserting into the last of the above equations:

$$\mu/s - \mu P_1^*(s) - \lambda P_1^*(s) = sP_1^*(s) - 1$$

which is solved wrt $P_1^*(s)$:

$$P_1^*(s) = \frac{1}{\lambda + \mu + s} + \frac{\mu}{s} \frac{1}{\lambda + \mu + s}$$

This expression is not recognized in the list of Laplace transforms. A trick is now to multiply with $(\lambda + \mu)/(\lambda + \mu)$:

$$P_1^*(s) = \frac{\lambda + \mu}{\lambda + \mu} \left(\frac{1}{\lambda + \mu + s} + \frac{\mu}{s} \frac{1}{\lambda + \mu + s} \right) =$$

$$\frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\lambda + \mu + s} + \frac{\lambda}{\lambda + \mu} \cdot \frac{\mu}{s} \cdot \frac{1}{\lambda + \mu + s} + \frac{\mu}{\lambda + \mu} \cdot \frac{1}{\lambda + \mu + s} + \frac{\mu}{\lambda + \mu} \cdot \frac{\mu}{s} \cdot \frac{1}{\lambda + \mu + s}$$

$$= \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\lambda + \mu + s} + \frac{\mu}{s} \cdot \frac{\lambda + s + \mu}{\lambda + \mu} \cdot \frac{1}{\lambda + \mu + s} = \frac{\lambda}{\lambda + \mu} \cdot \frac{1}{\lambda + \mu + s} + \frac{\mu}{\lambda + \mu} \cdot \frac{1}{s}$$

Now using $\mathscr{L}e^{\alpha t} = 1/(s-\alpha)$ and $\mathscr{L}1 = 1/s$ we find the inverse Laplace of $P_1^*(s)$ ($-\alpha = \lambda + \mu$):

$$P_1(t) = \frac{\lambda}{\lambda+\mu} e^{-(\lambda+\mu)t} + \frac{\mu}{\lambda+\mu}$$

and

$$P_0(t) = 1 - P_0(t) = \frac{\lambda}{\lambda+\mu}\left(1 - e^{-(\lambda+\mu)t}\right)$$

Note that when $t > 3/(\lambda+\mu)$ the time dependent solution is deviating from the steady state solution with only 5%. In practice, we therefore often say that steady state is achieved after 3 times the shortest expected transition time, here $3/\mu$. The time dependent solution is often needed in FTA and RBD analyses.