# TPK4120 - Lecture summary

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## Chapter 7 - Reliability Importance Metrics

In the textbook very many reliability importance metrics are presented. We only focus on the following::

- Birnbaum's measure*
- Improvement Potential*
- The criticality importance measure*
- Fussel-Vesley's measure*

### Why do we reliability importance metrics?

There are many reasons to investigate component importance:

- Considering improving the inherent reliability of critical components
- Establish a preventive maintenance program for the most critical components
- Ensure that we have sufficient spare parts for critical components
- Considering implementing (extra) redundancy at component level for the most critical components
- Given that we have a system failure, which component is the most likely to have caused this?

Several measures are discussed, and the various measures will have their strength and weakness to answer the questions above.

## Birnbaum's Metric of Reliability Importance

Birnbaum's metric of reliability importance of a component is a sensitivity measure expressing the change in system reliability if component $i$ is slightly changed, i.e.,;

$$I^{\mathrm{B}}(i \mid t) = \frac{\partial h(\mathbf{p}(t))}{\partial p_i(t)} = \frac{\partial Q_0(t)}{\partial q_i(t)}$$

It follows that a small change $\Delta p_i(t)$ in the component reliability will result in the following change in system reliability:

$$\Delta h(\mathbf{p}(t)) = I^{\mathrm{B}}(i \mid t)\Delta p_i(t)$$

A disadvantage with Birnbaum's metric is that it is difficult to calculate. If we are able to write down the system reliability function, it should be rather easy to find Birnbaum's measure. But in practice we will not be able to write down the system reliability function explicitly, and hence we cannot derive Birnbaum's metric. In some cases we may utilize that:

$$I^{\mathrm{B}}(i \mid t) = h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t))$$

which may be used if we are able to determine $h(\mathbf{p}(t))$ with reasonable high precision. The results follows from pivotal decomposition of $h(\mathbf{p}(t))$ around component $i$.

In the same way as $h(\mathbf{p}(t)) = \mathrm{E}(\phi(\mathbf{X}(t))$ we have that $h(\cdot_i, \mathbf{p}(t)) = \mathrm{E}(\phi(\cdot_i, \mathbf{X}(t))$, and hence from $I^{\mathrm{B}}(i \mid t) = h(1_i, \mathbf{p}(t)) - h(0_i, \mathbf{p}(t))$ it follows that

$$\begin{aligned} I^{\mathrm{B}}(i \mid t) &= \mathrm{E}(\phi(1_i, \mathbf{X}(t)) - \mathrm{E}(\phi(0_i, \mathbf{X}(t)) \\ &= \mathrm{E}\left[\phi(1_i, \mathbf{X}(t) - \phi(0_i, \mathbf{X}(t)\right] \\ &= \Pr(\phi(1_i, \mathbf{X}(t) - \phi(0_i, \mathbf{X}(t) = 1) \end{aligned}$$

$\phi(1_i, \mathbf{X}(t) - \phi(0_i, \mathbf{X}(t) = 1$ means that component $i$ is critical meaning that the state of component $i$ is decisive for whether or not the system functions. From this we can give an alternative definition of Birnbuam's metric:

$I^{\mathrm{B}}(i \mid t)$ is the probability that component $i$ is critical at time $t$.

### Birnbaum's measure for a series structure

For a series structure we have that $h(\mathbf{p}(t)) = \prod_j p_j(t)$. It then follows by taking the partial derivative that

$$I^{\mathrm{B}}(i \mid t) = \frac{\prod_j p_j(t)}{p_j(t)}$$

provided $p_j(t)$ is greater than zero. From this we see that the component with the *lowest* reliability in a series structure is the most critical one with respect to Birnbaum's measure. This corresponds to the saying that a chain is no stronger than the weakest link.

#### Birnbaum's measure for a parallel structure

For a parallel structure it follows similarly that:

$$I^{\mathrm{B}}(i \mid t) = \frac{1 - \prod_j (1 - p_j(t))}{1 - p_j(t)}$$

provided $p_j(t)$ is less than one. From this we see that the component with the highest reliability in a parallel structure is the most critical one with respect to Birnbaum's measure. This means that to improve the reliability of a parallel structure, we should improve the best component.

### Improvement Potential

The Iimprovement Potential states how much the system reliability will increase if component $i$ is replaced with a perfect component:

$$I^{\mathrm{IP}}(i \mid t) = h(1_i, \mathbf{p}(t)) - h(\mathbf{p}(t))$$

It is easy to show the following relation to Birnbaum's measure:

$$I^{\mathrm{IP}}(i \mid t) = I^{\mathrm{B}}(i \mid t)(1 - p_i(t)) = I^{\mathrm{B}}(i \mid t) q_i(t)$$

### Criticality Importance

The criticality importance measure $I^{\mathrm{CR}}(i \mid t)$ of component $i$ at time $t$ is the probability that component $i$ is critical for the system and is failed at time $t$, when we know that the system is failed at time $t$. It is easy to show the following relation to Birnbaum's measure:

$$I^{\mathrm{CR}}(i \mid t) = \frac{I^{\mathrm{B}}(i \mid t) \cdot (1 - p_i(t))}{1 - h(\mathbf{p}(t))} = \frac{I^{\mathrm{B}}(i \mid t) \cdot q_i(t)}{Q_0(t)}$$

### Fussell-Vesely's Metric

The Fussell-Vesely's importance metric $I^{\mathrm{FV}}(i \mid t)$ of component $i$ at time $t$ is the probability that at least one minimal cut set that contains component $i$ is failed at time $t$, when we know that the system is failed at time $t$.

In order to calculate $I^{\mathrm{VF}}(i \mid t)$ we need some reasoning. We simplify and skip the index $t$. Now introduce the following notation (we use the terminology "component" whereas the precise word would be "basic event"):

- $D_i$: At least one minimal cut containing component $i$ is failed
- $C$: The system is failed
- $m_i$: Number of minimal cut set containing component $i$
- $E_j^i$: Minimal cut set $j$ containing component $i$ is failed

From the definition we have:

$$I^{\mathrm{FV}}(i) = \Pr(D_i \mid C) = \frac{\Pr(D_i \cap C)}{\Pr(C)}$$

Since $D_i$ is a subset $C$, then $D_i \cap C = D_i$ and we have:

$$I^{\mathrm{FV}}(i) = \frac{\Pr(D_i)}{\Pr(C)}$$

To find $\Pr(D_i)$ we use the same approach as for the "upper bound' approximation for $Q_0$. However, note that $D_i = E_1^i \cup E_2^i \cup \cdots \cup E_{m_i}^i$ where the union is only taken over minimal cut sets containing component $i$. This gives:

$$\Pr(D_i) = 1 - \Pr(E_1^{i\,C} \cap E_1^{2\,C} \cap \cdots \cap E_{m_i}^{i\;C}) \le 1 - \Pr(E_1^{i\,C})\Pr(E_1^{2\,C}) \cdots \Pr(E_{m_i}^{i\;C})$$

$\Pr(E_j^{i\,C})$ is then obtained by one minus the probability for the event that minimal cut set $j$ is failed, i.e., $\Pr(E_j^{i\,C}) = 1 - \check{Q}_j^i = 1 - \prod_{l \in K_j} q_l$. The following approximation is usually sufficient to calculate Fussell-Vesely's measure:

$$I^{\mathrm{FV}}(i) \approx \frac{1 - \prod_j^{m_i}(1 - \check{Q}_j^i)}{Q_0}$$

where the product is over minimal cut sets which contain component $i$.

If cut set failure probabilities are small, a faster approximation is given by:

$$I^{\mathrm{FV}}(i) \approx \frac{\sum_j^{m_i} \check{Q}_j^i}{Q_0}$$

where the sum is over minimal cut sets which contain component $i$.

By comparing the definition of $I^{\mathrm{CR}}(i)$ and $I^{\mathrm{FV}}(i)$, we see that these measures are rahter close to each other. Thus by assuming $I^{\mathrm{CR}}(i) \approx I^{\mathrm{FV}}(i)$, we could easily get an approximation of Birnbaum's measure from:

$$I^{\mathrm{B}}(i) = \frac{I^{\mathrm{CR}}(i) \cdot Q_0}{q_i} \approx \frac{I^{\mathrm{VF}}(i) \cdot Q_0}{q_i}$$

## System failure frequency (ROCOF)

In Chapter 6 various methods were presented to calculate system reliability ($p_S$), TOP event probabilities ($Q_0$) and other metrics. These measures are important because they can be used to calculate production unavailability costs. In some situations there will also be cost related to the *event* that we have a system failure. This is the case also when we consider accidents, it is not the average time we have an accident which is of interest, but how often an accident occurs. This calls for a system failure frequency measure.

One way to calculate system failure frequency, $F_0$, is to start with Birnbaum's measure. First we recall that $I^{\mathrm{B}}(i)$ is the probability that the system is in such a state that component $i$ is critical. That a component is critical means that the system is in such a state that the system is functioning if component $i$ is functioning, and in a fault state if component $i$ is failed. Then it follows that:

$$w_S = \sum_i I^{\mathrm{B}}(i) p_i \lambda_i = \sum_i I^{\mathrm{B}}(i)(1 - q_i)\lambda_i$$

where $p_i = 1 - q_i$ is the probability that component $i$ is functioning, and $\lambda_i$ is the failure rate of component $i$. Thus, the contribution of component $i$ to $F_0$ is given as the product of:

- The probability that component $i$ is critical, i.e., the state of other components
- The probability that component $i$ is functioning
- The failure rate of component $i$

### Maintenance consideration

Birnbaum's measure is also relevant for maintenance consideration. $I^{\mathrm{B}}(i)q_i$ could be seen as the system unavailability "caused" by component $i$. Thus multiplying production downtime cost per hour by $I^{\mathrm{B}}(i)q_i$ yields the unavailability costs caused by component $i$. Thus $I^{\mathrm{B}}(i)q_i$ could be used to screen components for inclusion in a preventive maintenance program. A follow up question is then what is the downtime cost as a function of the failure rate of the component, having in mind that a preventive maintenance program is aimed to reduce the failure rate. If $C_{\mathrm{U}}$ is the unavailability cost per time unit, then the failure cost per time unit attributed to component $i$ is given by

$$C_i(\lambda_i) = \lambda_i C_{\mathrm{U}} I^{\mathrm{B}}(i) p_i \approx \lambda_i C_{\mathrm{U}} I^{\mathrm{B}}(i)$$

### Final remarks / short summary

- Birnbaum's measure and the criticality importance measure are the most intuitive measures
- If we have an easy formula for the system reliability, $p_S = h(\mathbf{p})$, it is straight forward to calculate $I^{\mathrm{B}}(i) = \frac{\partial h(\mathbf{p})}{\partial p_i}$ and $I^{\mathrm{CR}}(i) = \frac{I^{\mathrm{B}}(i)(1-p_i)}{1-h(\mathbf{p})}$
- These two measures are often difficult to calculate, hence Fussell-Vesely's measure is often considered due to the easy way to calculate
- The formula $I^{\mathrm{FV}}(i) \approx \frac{1-\prod_j^{m_i}(1-\check{Q}_j^i)}{Q_0}$ is essential to understand

- We may then use: $I^{\mathrm{B}}(i) = \frac{I^{\mathrm{CR}}(i) Q_0}{q_i(t)} \approx \frac{I^{\mathrm{FV}} Q_0(t)}{q_i}$
- Birnbaum's measure can be used both to find system failure frequency and cost contribution for the various components.