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RELIABILITY AND MAINTENANCE

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Introduction

- Reliability and maintenance of a production line is essential for modern production
- Objectives for these lectures on reliability and maintenance, TPK4191:
  - Understand simple reliability modelling
  - Introduction to maintenance modelling
  - Digital twins and synchronization of maintenance and operations

Definitions

- **Reliability**: The ability of an item to perform a required function, under given environmental and operational conditions and for a stated period of time
- **Maintenance**: All actions necessary for retaining an item in or restoring it to a specified condition. Maintenance can be corrective or preventive.
- **Availability**: The ability of an item (under combined aspects of its reliability, maintainability, and maintenance support) to perform its required function at a stated instant of time or over a stated period of time
- Availability at time t: $A(t) = \Pr(\text{item is functioning at time } t)$

Reliability terminology

In the following T is a stochastic variable representing the time-to-failure of a component:

- Distribution function: $F(t) = \Pr(T \leq t)$ (=CDF)
- Probability density function: $f(t) = \frac{d}{\mathrm{d}t}F(t)$ (=PDF)
- Survivor function: $R(t) = \Pr(T > t) = 1 - F(t)$
- Failure rate function: $z(t) = \frac{f(t)}{R(t)}$
- Mean time to failure: $\text{MTTF} = \int_0^\infty t \cdot f(t)\,\mathrm{d}t = \int_0^\infty R(t)\,\mathrm{d}t$
- Mean down time: $\text{MDT} = \int_0^\infty t \cdot f_D(t)\,\mathrm{d}t$ (where $f_D(t)$ is the distribution of the downtime D)

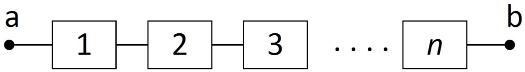
Conditional quantities

- The failure rate function, often denoted the hazard rate, is the conditional probability that a component that has survived t will fail in a small time interval $[t, t+\Delta t>$
- This means that $\Delta t \cdot z(t) \approx \Pr(t < T \leq t+\Delta t | T > t)$
- To express the *conditional* survivor function we introduce $R(x|t) = \Pr(T > x+t | T > t)$, i.e., the conditional probability of surviving another x time units given that the component has survived t time units:

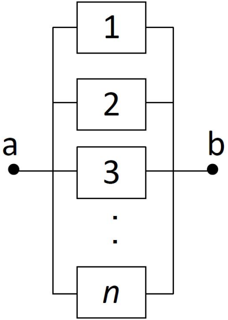
$$R(x|t) = \frac{\Pr(T > x+t \cap T > t)}{\Pr(T > t)} = \frac{\Pr(T > x+t)}{\Pr(T > t)} = \frac{R(t+x)}{R(t)}$$

Reliability Block Diagram (RBD)

- Reliability block diagrams are valuable when we want to visualise the performance of a system comprised of several (binary) components.
- Interpretation of the RBD:
  - The system is functioning if there is a connection between *a* and *b*, i.e., it is a path of functioning components from *a* to *b*
  - The system is in a fault state (is not functioning) if it does not exist a path of functioning components between *a* and *b*
- Example of a series structure in a RBD:



Parallel structure



State variable

The state variable for a component is given by:

$$x_i(t) = \begin{cases} 1 \text{ if component } i \text{ is functioning at time } t \\ 0 \text{ if component } i \text{ is in a fault state at time } t \end{cases}$$

Structure function

For the system we now introduce

$$\phi(\mathbf{x}, t) = \begin{cases} 1 \text{ if the system is functioning at time } t \\ 0 \text{ if the system is in a fault state (not functioning) at time } t \end{cases}$$

- ϕ denotes the structure function, and depends on the x_is (**x** is a vector of all the x_is)
- $\phi(\mathbf{x}, t)$ is thus a mathematical function that uniquely determines whether the system functions or not for a given value of the **x**-vector

The structure function for some simple structures

For a series structure we have

$$\phi(\mathbf{x}) = x_1 \cdot x_2 \cdot \ldots \cdot x_n = \prod_{i=1}^{n} x_i \tag{1}$$

For a parallel structure we have

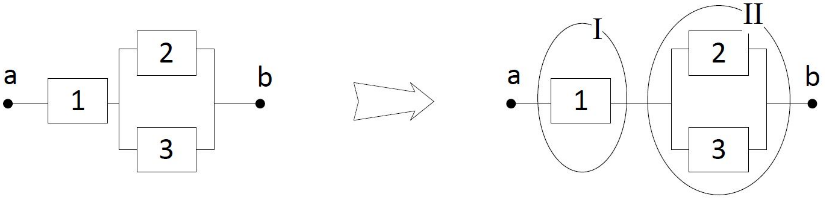
$$\phi(\mathbf{x}) = 1 - (1 - x_1)(1 - x_2) \ldots (1 - x_n) = 1 - \prod_{i=1}^{n} (1 - x_i) = \coprod_{i=1}^{n} x_i$$

Note that we for two components may simplify:

$$\phi(x_1, x_2) = x_1 \amalg x_2 = 1 - (1 - x_1)(1 - x_2) = x_1 + x_2 - x_1 x_2 \tag{2}$$

Composed structures

For structures comprised of series and parallel structures we may combine the above formulas by splitting the reliability block diagram into sub-blocks, and then apply the formulas within a block, and then treat a sub-block as a block on a higher level:



- The RBD is split into sub-blocks, here I and II
- We may then write $\phi(\mathbf{x}) = \phi_{\mathrm{I}} \times \phi_{\mathrm{II}}$ because I and II are in series
- Further, $\phi_{\mathrm{I}} = x_1$ and $\phi_{\mathrm{II}} = x_2 + x_3 - x_2x_3 \implies \phi(\mathbf{x}) = x_1x_2 + x_1x_3 - x_1x_2x_3$

System reliability for independent components

- Up to now we have used the symbol x_i to represent the value a state variable may take
- In order to assess component and system reliability, we need to treat the state variables and the structure function as stochastic variables
- Let $X_i(t)$ denote the state variable i, and $\mathbf{X}(t) = (X_1(t), X_2(t), \ldots X_n(t))$ be the state vector
- $\implies$ the structure function is now a stochastic variable $\phi(\mathbf{X}(t))$

System reliability for independent components, cont.

Introduce the following probabilities:

$$p_i(t) = \Pr(X_i(t) = 1) = \text{ Component reliability}$$
$$p_S(t) = \Pr(\phi(\mathbf{X}(t)) = 1) = \text{ System reliability}$$

Since both the state variables and the structure function is binary we have:

$$E(X_i(t)) = p_i(t)$$
$$E(\phi(\mathbf{X}(t))) = p_S(t)$$

System reliability for independent components, cont.

- Assume that we are able to write the structure function as a sum of products of the state variables
  - E.g., $\phi(\mathbf{x}) = x_1 x_2 + x_1 x_3 - x_1 x_2 x_3$
- We then use the results that "the expectation of a sum equals to the sum of expectations" and "the expectation of a product equals the product of the expectations" (assuming independence) $\implies$
- $p_S = \mathrm{E}(\phi(\mathbf{X}))$ = the sum of products of expectations, i.e., a sum of products of $\mathrm{E}(X_i)$'s (=p_i's)
  - E.g., $p_S = \mathrm{E}(X_1 X_2 + X_1 X_3 - X_1 X_2 X_3) =$
    $\mathrm{E}(X_1)\mathrm{E}(X_2) + \mathrm{E}(X_1)\mathrm{E}(X_3) - \mathrm{E}(X_1)\mathrm{E}(X_2)\mathrm{E}(X_3) = p_1 p_2 + p_1 p_3 - p_1 p_2 p_3$
- That is, to find $p_S(t)$ replace all the x_i's in the structure function with corresponding p_i's

Component reliability

- Component reliability depends on maintenance and operations strategies
- In most situations it is rather straight forward to argue that if reliability, *p*, is measured by availability, *A*, we have:

$$p_i = \frac{\mathrm{MTTF}_i}{\mathrm{MTTF}_i + \mathrm{MDT}_i}$$

- Where MDT_i is the mean down time, and MTTF_i is the mean time to failure of component *i*
- Note
  - MTTF_i depends on the maintenance strategy
  - MDT_i depends ons spare part availability, maintainability etc.

Exam 2017 1c)

- A separate factory is used for production of raw material I (from the SIMPLEX!)
- The production is essential a two step production where we have machine 1 and machine 2 in a series structure
- Mean time to failure (MTTF) and mean down time (MDT) upon a failure for the first machine are 500 hours and 50 hours respectively
- For the second machine these numbers are given by 400 hours and 50 hours respectively
- Find the system reliability p_S for the series structure of machine 1 and machine 2

Exam 2017 1c), cont

- Assume that production of raw material I is proportional to the system reliability, and that a production of 100 units per hour of raw material I (current situation) applies for configuration of machine 1 and machine 2 in series
- Calculate the production of raw material I per hour if a redundant machine 2 is installed, i.e., machine 1 is in series with two machines of type 2 in parallel (i.e., a redundant machine 2)
- How much could we pay per hour for this redundant machine? Hint: Calculate the increase in production of raw material I, and use the result from 1b)

Solution

Machine	MTTF	MDT	A = MTTF/(MTTF+MDT)
1	500	50	0.909091
2	400	50	0.888889

- Component reliability, p_i, is measured in terms of availability, *A*
- System reliability for a series structure is given by:
- $p_S = p_1 p_2 \approx 0.909091 \cdot 0.888889 \approx 0.808080808 \approx 81\%$

Solution

- Let maximum production capacity is X units per hour, and current production = 100 = $p_S X \implies X = 100/p_S \approx 123.75$
- By installing a redundant machine 3 in parallel with machine 2, system reliability is given by ($p_2 = p_3$):

$$p_S = p_1(p_2 + p_3 - p_2 p_3) = p_1(2p_2 - p_2^2) = 0.897868$$

- $\implies$ new production = $p_S X \approx 111.11$
- Compared to the current situation this is an increase of 11.11 units
- The marginal profit increase by one unit of raw material I is NOK 5 (from the shadow price in problem 1b), hence the redundant machine gives a total increase of profit equal to NOK 55.55 per hour
- Hence, we could pay NOK 55 per hour (=487 000 per year) to defend the investment

Component importance

- In many situations we would like to known the most important components?
  - Which components to prioritize wrt maintenance
  - Components where we would like to consider redundancy
  - Components where we are willing to pay a higher cost for higher reliability
- Different component importance measure exist
- We only consider Birnbaums measure

Birnbaums measure of reliability importance

- Introduce $h(\mathbf{p}) = p_S(t) = \Pr(\phi(\mathbf{X}(t)) = 1)$
- Birnbaums measure of reliability importance is given by

$$I^{\mathrm{B}}(i) = \frac{\partial h(\mathbf{p})}{\partial p_i}$$

- It follows that a small change $\Delta p_i(t)$ in the component reliability will result in the following change in system reliability:

$$\Delta h(\mathbf{p}) = I^{\mathrm{B}}(i)\Delta p_i$$

- Since $h(\mathbf{p})$ is linear in each p_i it follows that:

$$I^{\mathrm{B}}(i) = h(\mathbf{p} \mid p_i = 1) - h(\mathbf{p} \mid p_i = 0)$$

Interpretation of $I^{\mathrm{B}}(i)$

- From

$$\begin{aligned} I^{\mathrm{B}}(i) &= h(\mathbf{p} \mid p_i = 1) - h(\mathbf{p} \mid p_i = 0) \\ &= \Pr[\phi(\mathbf{X} \mid X_i = 1) = 1] - \Pr[\phi(\mathbf{X} \mid X_i = 0) = 1] \\ &= \mathrm{E}[\phi(\mathbf{X} \mid X_i = 1)] - \mathrm{E}[\phi(\mathbf{X} \mid X_i = 0)] \\ &= \mathrm{E}[\phi(\mathbf{X} \mid X_i = 1) - \phi(\mathbf{X} \mid X_i = 0)] \\ &= \Pr[\phi(\mathbf{X} \mid X_i = 1) - \phi(\mathbf{X} \mid X_i = 0) = 1] \\ &= \Pr[\text{Component } i \text{ makes the difference}] \end{aligned}$$

- it follows that $I^{\mathrm{B}}(i)$ is the probability that component i is critical, i.e., the system is functioning if and only if component i is functioning

Maintenance model - Single component considerations

- Consider a component where a preventive maintenance (PM) action is conducted at predetermined intervals due to an increasing failure rate function $z(t)$
- Typically we assume that failure times are Weibull distributed where the failure rate function is given by

$$z(t) = \alpha\lambda^{\alpha} t^{\alpha-1} \tag{3}$$

- In order to find an optimal interval for a maintenance action we may establish the expected cost per time unit as a function of the maintenance interval, say τ:

Expected cost per time unit

$$C(\tau) = c_{\text{PM}}/\tau + \lambda_{\text{E}}(\tau)\,[c_{\text{CM}} + c_{\text{EP}} + c_{\text{ES}}]$$

where

- c_{PM} is the cost of a preventive maintenance action
- $\lambda_{\text{E}}(\tau)$ is the *effective failure rate*, i.e., the expected number of failures per time unit when the component is preventively maintained every τ time unit
- c_{CM} is the cost of a corrective maintenance (CM) action
- c_{EP} is the expected production losses upon a component failure
- c_{ES} is the expected safety cost upon a component failure, including material damages and environmental losses
- In the following we let $c_U = [c_{\text{CM}} + c_{\text{EP}} + c_{\text{ES}}]$

$\lambda_E(\tau)$

- The Weibull distribution is a widely used distribution for ageing components
- In the case of Weibull distributed life times we may find approximation formulas for the effective failure rate:
- Let MTTF be the mean time to failure without maintenance, and α be the ageing parameter of the lifetime distribution of the component
- An approximation of the effective failure rate is:

$$\lambda_E(\tau) = \left(\frac{\Gamma(1 + 1/\alpha)}{\mathrm{MTTF}}\right)^{\alpha} \tau^{\alpha - 1}$$

Optimal value of τ

By setting the derivative of the cost equation (objective function) equal to zero, we find the optimal interval:

$$\tau^* = \frac{\mathrm{MTTF}}{\Gamma(1 + 1/\alpha)} \left(\frac{c_{\mathrm{PM}}}{c_{\mathrm{U}}(\alpha - 1)} \right)^{1/\alpha}$$

Example

- In a production line a packing machine is critical for the production
- $\mathrm{MTTF} = 600$ hours ($\approx$ one month) if the machine is not preventively maintained
- Failure times are assumed to be Weibull distributed with ageing parameter $\alpha = 4$
- The cost of a preventive maintenance activity is $c_{\mathrm{PM}} = 5\,000$ NOKs
- If the machine fails the total cost of corrective maintenance and lost production is $c_{\mathrm{U}} = 35\,000$
- Assume "24/7"

$$\tau^* = \frac{\mathrm{MTTF}}{\Gamma(1+1/\alpha)}\left(\frac{c_{\mathrm{PM}}}{c_{\mathrm{U}}(\alpha-1)}\right)^{1/\alpha} \approx \frac{600}{0.906}\left(\frac{5000}{35000\times 3}\right)^{1/4} \approx 309 \text{ hours}$$

‣ Show in Excel

λ with and without maintenance

- The baseline formula for the effective failure rate is given by:

$$\lambda_{\mathrm{E}}(\tau) = \left(\frac{\Gamma(1 + 1/\alpha)}{\mathrm{MTTF_N}}\right)^{\alpha} \tau^{\alpha-1}$$

- where $\mathrm{MTTF_N}$ is the “naked” MTTF, i.e., without maintenance
- Now, assume that $\lambda_{\mathrm{E}}(\tau_0) = \lambda_0$ is the effective failure rate with the current maintenance interval, τ_0
- It then follows that

$$\lambda_{\mathrm{E}}(\tau) = \lambda_0 \left(\frac{\tau}{\tau_0}\right)^{\alpha-1}$$

MTTF with and without maintenance

- In the same way it the follows that

$$\mathrm{MTTF}_{\mathrm{E}}(\tau) = \mathrm{MTTF}_0 \left(\frac{\tau_0}{\tau}\right)^{\alpha-1}$$

- where MTTF_0 is the MTTF with the current maintenance interval, i.e., τ_0
- In the availability formula we use $\mathrm{MTTF}_{\mathrm{E}}(\tau)$, i.e.,:

$$A = \frac{\mathrm{MTTF}_{\mathrm{E}}(\tau)}{\mathrm{MTTF}_{\mathrm{E}}(\tau) + \mathrm{MDT}}$$

Exam 2017 1c, extended

- Assume that the MTTT for machine 2 is the effective MTTF given the current maintenance interval $\tau_0 = 40$, i.e., $\mathrm{MTTF}_\mathrm{E}(40) = 400$
- The value of one hour production is $c_\mathrm{P} = 123.75 \cdot 5 = 618.75$
- Assume that $c_\mathrm{PM} = 5000$ and $c_\mathrm{CM} = 20000$
- Find the optimal maintenance interval for the new redundant machine of type 2

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Thank you for your attention

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