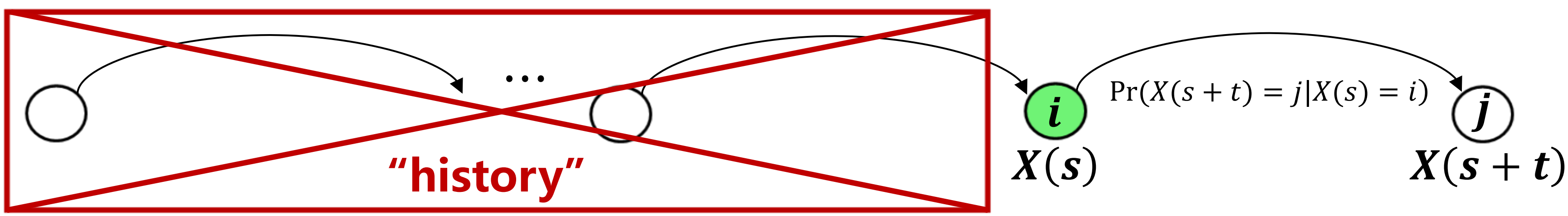
Markov property and Markov process

Definition 1: A process is said to have Markov property if

$$\boxed{\Pr(X(s+t)=j|X(s)=i, X(u)=x(u), 0 \le u < s)} = \boxed{\Pr(X(s+t)=j|X(s)=i)}$$

for all $x(u), 0 \le u < s$



If the present state of the process is known, the future state of the process is independent of anything happened in the past.

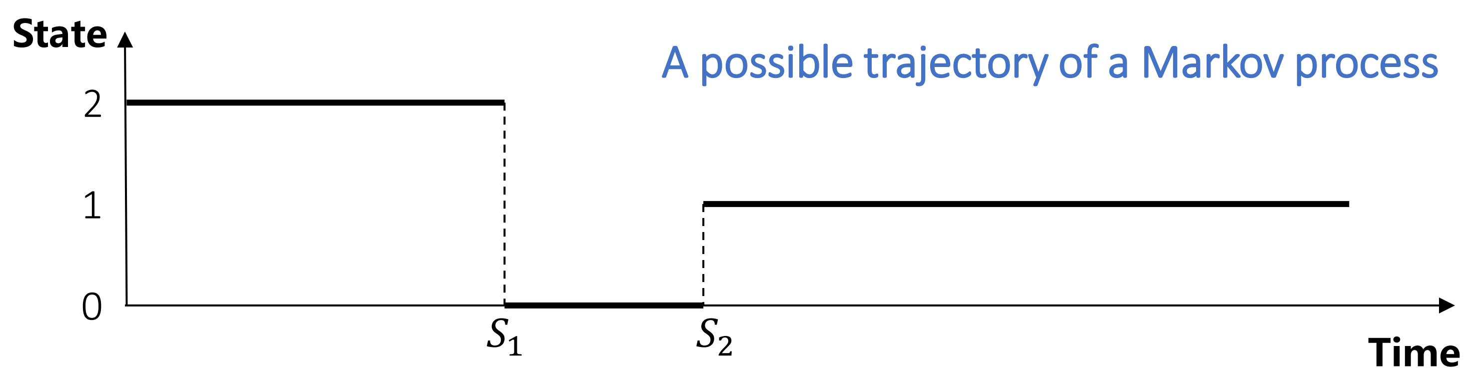
Recall: Transition probabilities

Assume that the Markov process is a process fulfills for all i, j in $\mathcal{X} = \{0, 1, 2, \dots, r\}$

$$\Pr(X(s+t) = j | X(s) = i) = \Pr(X(t) = j | X(0) = i) \quad \text{for all} \quad s, t \geq 0$$

probability of a transition from state i to state j

- **Independent of global time**
- **Depends only on time interval available for the transition**



$\tilde{T}_i$ is the sojourn time in state i. (the time spent during a visit to state i)

$\tilde{T}_i$ is memoryless and must be exponentially distributed.

Recall: Transition rates

The transition rate is defined as

$$a_{ij} = \alpha_i \cdot P_{ij} \qquad \textbf{for all} \qquad i \neq j$$

Since $\boldsymbol{\alpha_i}$ is the rate at which the process leaves state $\boldsymbol{i}$ and $\boldsymbol{P_{ij}}$ is the probability that it goes to state $\boldsymbol{j}$, it follows that $\boldsymbol{a_{ij}}$ is the rate when in state $\boldsymbol{i}$ that the process makes a transition into state $\boldsymbol{j}$.

Since $\sum_{j\neq i} P_{ij} = 1$, it holds that

$$\alpha_i = \sum_{\substack{j=0 \\ j\neq i}}^{r} a_{ij}$$

Let $\boldsymbol{T_{ij}}$ be **the time spent during a visit from state** $\boldsymbol{i}$ to $\boldsymbol{j}(\neq \boldsymbol{i})$, then $\boldsymbol{T_{ij}}$ is **exponentially distributed with rate** $\boldsymbol{a_{ij}}$

Recall: Transition rates matrix

Arrange the transition rate $\boldsymbol{a_{ij}}$ as a matrix:

$$\mathbb{A} = \begin{pmatrix} \boxed{a_{00}} & a_{01} & \cdots & a_{0r} \\ a_{10} & \boxed{a_{11}} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r0} & a_{r1} & \cdots & \boxed{a_{rr}} \end{pmatrix} \qquad a_{ii} = -\alpha_i = -\sum_{\substack{j=0 \\ j \neq i}}^{r} a_{ij}$$

- Entries of row $\boldsymbol{i}$ are the **departure rates from state $\boldsymbol{i}$** and the opposite number of the diagonal element $-\boldsymbol{a_{ii}} = \boldsymbol{\alpha_i}$ is the total departure rate from state $\boldsymbol{i}$. Note that:
  - $\boldsymbol{\alpha_i}$ is also the rate of staying in **state $\boldsymbol{i}$**.
  - $\boldsymbol{a_{ij}}$ is the rate of leaving from **state $\boldsymbol{i}$** to **state $\boldsymbol{j}$**.
- The sum of the entries in row $\boldsymbol{i}$ **is always equal to 0, for all $\boldsymbol{i} \in \mathcal{X}$.**

Recall: State equations

According to the Kolmogorov forward equations :

$$\dot{P}_j(t) = \sum_{k=0}^{r} a_{kj} P_k(t)$$

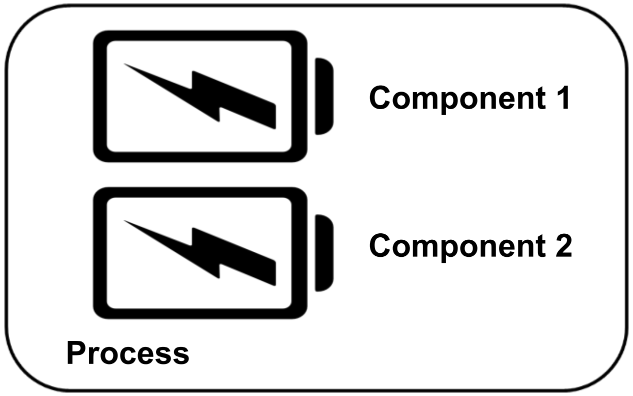
In matrix term, the state equations for Markov process is written as :

$$[P_0(t), \dots, P_r(t)] \cdot \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0r} \\ a_{10} & a_{11} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r0} & a_{r1} & \cdots & a_{rr} \end{pmatrix} = [\dot{P}_0(t), \dots, \dot{P}_r(t)]$$

or in compact form $\mathbf{P}(t) \cdot \mathbb{A} = \dot{\mathbf{P}}(t)$

Example 1

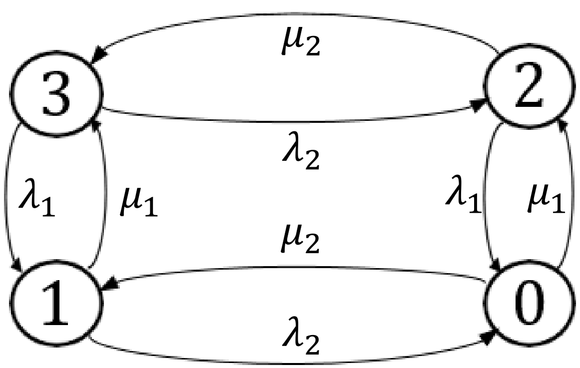
How to describe the transition of states ?



State	Component 1	Component 2
3	Functioning	Functioning
2	Functioning	Failed
1	Failed	Functioning
0	Failed	Failed

Assume that:

- Each unit has its own repairer. A repair action brings the unit back to functioning
- The units have constant failure rates $\boldsymbol{\lambda_i}$ and constant repair rates $\boldsymbol{\mu_i, i = 1, 2}$
- We only consider the events of single transition

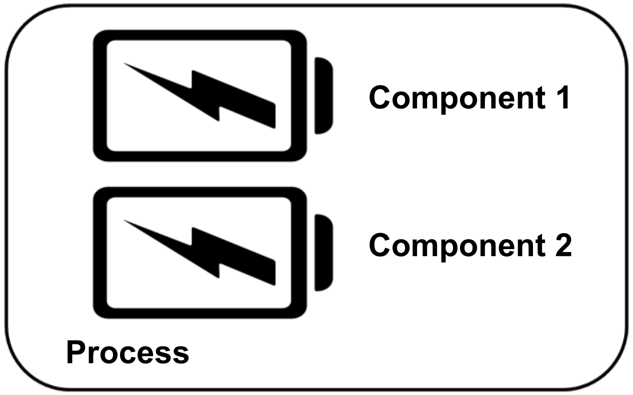


$$\mathbb{A} = \begin{pmatrix} -(\mu_2+\mu_1) & \mu_2 & \mu_1 & 0 \\ \lambda_2 & -(\lambda_2+\mu_1) & 0 & \mu_1 \\ \lambda_1 & 0 & -(\lambda_1+\mu_2) & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\lambda_1+\lambda_2) \end{pmatrix}$$

Recall: How to describe the transition of states? II

Example 1

How to calculate the Mean Downing Time (MDT) ?



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Assume that:

- Each unit has its own repairer. A repair action brings the unit back to functioning
- The units have constant failure rates $\boldsymbol{\lambda_i}$ and constant repair rates $\boldsymbol{\mu_i, i = 1, 2}$
- We only consider the events of single transition

$$\mathbb{A} = \begin{pmatrix} -(\mu_2+\mu_1) & \mu_2 & \mu_1 & 0 \\ \lambda_2 & -(\lambda_2+\mu_1) & 0 & \mu_1 \\ \lambda_1 & 0 & -(\lambda_1+\mu_2) & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\lambda_1+\lambda_2) \end{pmatrix}$$

$$\alpha_0 = \mu_1 + \mu_2$$

departure rate from state 0

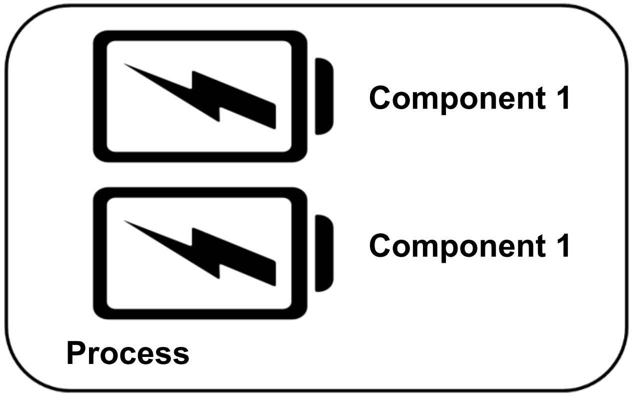
$$E\left(\tilde{T}_0\right) = \frac{1}{\mu_1 + \mu_2}$$

Mean Downing Time (MDT): Sojourn time of state 0

Recall: How to describe the transition of states? III

Example 2

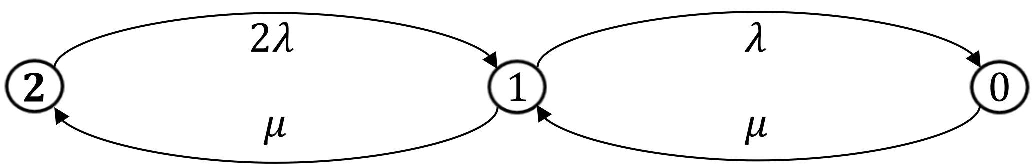
If the two components are independent and identical



State	Description
2	Both components are functioning
1	One component is functioning, and one is failed
0	Both components are in a failed state

Assume that:

- We have only a single repair crew that has adopted a first-fail-first-repair policy
- The units have the same constant failure rate $\boldsymbol{\lambda}$ and constant repair rate $\boldsymbol{\mu}$
- We only consider the events of single transition



$$\mathbb{A} = \begin{pmatrix} -\mu & \mu & 0 \\ \lambda & -(\lambda+\mu) & \mu \\ 0 & 2\lambda & -2\lambda \end{pmatrix}$$

The mean sojourn times in the three states are

$$E(\tilde{T}_2) = \frac{1}{2\lambda} \qquad E(\tilde{T}_1) = \frac{1}{\lambda+\mu} \qquad \boxed{E(\tilde{T}_0) = \frac{1}{\mu}}$$

MDT: **Sojourn time of state 0**

Markov Process: Time dependent solution

According to the state equations, it follows

$$\mathbf{P}(t) = \mathbf{P}(0)e^{t\mathbf{A}}$$

Although this is a very elegant solution, it is not very attractive since taking the exponential of a matrix is not that easy. Rewrite state equation as

$$\dot{\mathbf{P}}(t) = \lim_{\Delta t \to 0} \frac{\mathbf{P}(t+\Delta t) - \mathbf{P}(t)}{\Delta t} = \mathbf{P}(t)\mathbf{A}$$

$$\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)[\mathbf{A}\Delta t + \mathbf{I}]$$

$$\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)\left[\frac{\mathbf{A}\Delta t}{2} + \mathbf{I}\right]\left[\frac{\mathbf{A}\Delta t}{2} + \mathbf{I}\right]$$

$$\mathbf{P}(t+\Delta t) \approx \mathbf{P}(t)\left[\frac{\mathbf{A}\Delta t}{2^n} + \mathbf{I}\right]^{2^n}$$ calculation efficiency

Markov Process: Steady state solution

$\dot{\mathbf{P}}(t) \to 0$ when $t \to \infty$, hence $\mathbf{P}(t)\mathbf{A} = 0$. Note that $\mathbf{A}$ is not full rank due to definition.

$$[P_0, P_1, \dots, P_r] \cdot \begin{pmatrix} a_{00} & a_{01} & \cdots & a_{0r} \\ a_{10} & a_{11} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ a_{r0} & a_{r1} & \cdots & a_{rr} \end{pmatrix} = [0,0, \dots, 0]$$

Replace it by $\sum_{j=0}^{r} P_j = 1$, we have

$$[P_0, P_1, \dots, P_r] \cdot \begin{pmatrix} 1 & a_{01} & \cdots & a_{0r} \\ 1 & a_{11} & \cdots & a_{1r} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & a_{r1} & \cdots & a_{rr} \end{pmatrix} = [1,0, \dots, 0]$$

Example 3

How to calculate the time dependent probabilities?

The main benefit of time-dependent probabilities is that we can study how the probabilities (e.g., availability) change with time.

To find the time-dependent probabilities for a specific transition model, we have to:

- Step1: Set up the transition matrix
- Step2: Set up the differential equations (state equations)
- Step3: Solve these equations

Laplace transform I

Laplace transform

- is an integral transform that converts a function of a real variable *t* (often time) to a function of a complex variable *s* (complex frequency).

- The transform has many applications in science and engineering because it is a tool for solving differential equations.

$$f^*(s) = \mathcal{L}\,[f(t)] = \int_0^\infty e^{-st} f(t)dt$$
$$\mathcal{L}\,[f'(t)] = s\,\mathcal{L}\,[f(t)] - f(0) = sf^*(s) - f(0)$$
$$\mathcal{L}\,[e^{at}] = 1/(s-a)$$
$$\mathcal{L}\,[1] = 1/s$$

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Laplace transform II

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
unit step	$u(t)$	$\frac{1}{s}$
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$
ramp	$t \cdot u(t)$	$\frac{1}{s^2}$
nth power (for integer n)	$t^n \cdot u(t)$	$\frac{n!}{s^{n+1}}$
qth power (for complex q)	$t^q \cdot u(t)$	$\frac{\Gamma(q+1)}{s^{q+1}}$
nth root	$\sqrt[n]{t} \cdot u(t)$	$\frac{1}{s^{\frac{1}{n}+1}}\Gamma\left(\frac{1}{n}+1\right)$
nth power with frequency shift	$t^n e^{-\alpha t} \cdot u(t)$	$\frac{n!}{(s+\alpha)^{n+1}}$
delayed nth power with frequency shift	$(t-\tau)^n e^{-\alpha(t-\tau)} \cdot u(t-\tau)$	$\frac{n! \cdot e^{-\tau s}}{(s+\alpha)^{n+1}}$
exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$

Function	Time domain $f(t) = \mathcal{L}^{-1}\{F(s)\}$	Laplace s-domain $F(s) = \mathcal{L}\{f(t)\}$
unit impulse	$\delta(t)$	1
delayed impulse	$\delta(t-\tau)$	$e^{-\tau s}$
unit step	$u(t)$	$\frac{1}{s}$
delayed unit step	$u(t-\tau)$	$\frac{1}{s}e^{-\tau s}$
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exponential decay	$e^{-\alpha t} \cdot u(t)$	$\frac{1}{s+\alpha}$

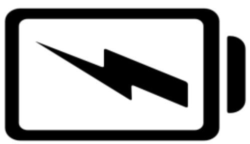
Selected Laplace transform: https://en.wikipedia.org/wiki/Laplace_transform

Markov Process: How to calculate time dependence probabilities? I

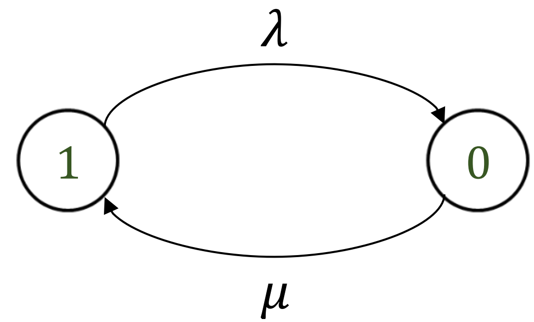
Example 3

How to calculate the time dependent probabilities?

Component 1



State	Component 1
1	Functioning
0	Failed



Assume that:

➢ The component is functioning at time 0, i.e., $P_1(0) = 1$

the transition matrix:

$$\mathbb{A} = \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix}$$

the state equation:

$$[P_0(t), P_1(t)] \cdot \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} = [\dot{P}_0(t), \dot{P}_1(t)]$$

with initial state

$$P_0(0) = 0, P_1(0) = 1$$

Example 3

How to calculate the time dependent probabilities?

Laplace transform:

$$f^*(s) = \mathcal{L}\,[f(t)] = \int_0^\infty e^{-st} f(t)dt$$

$$\mathcal{L}\,[f'(t)] = s\,\mathcal{L}\,[f(t)] - f(0) = sf^*(s) - f(0)$$

$$[P_0^*(s), P_1^*(s)] \cdot \begin{pmatrix} -\mu & \mu \\ \lambda & -\lambda \end{pmatrix} = [sP_0^*(s), sP_1^*(s) - 1]$$

$$P_1^*(s) = \frac{1}{\mu+\lambda+s} + \frac{\mu}{s}\frac{1}{\mu+\lambda+s}$$

the solution:

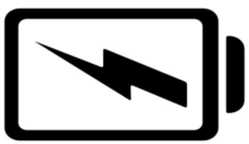
$$P_0(t) = \frac{\lambda}{\mu+\lambda} - \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t} \qquad P_1(t) = \frac{\mu}{\mu+\lambda} + \frac{\lambda}{\mu+\lambda} e^{-(\mu+\lambda)t}$$

$P_1(t)$ is the availability of the component

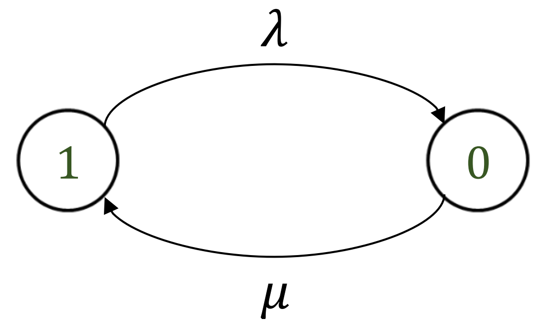
Example 3

How to calculate the time dependent probabilities?

Component 1



State	Component 1
1	Functioning
0	Failed



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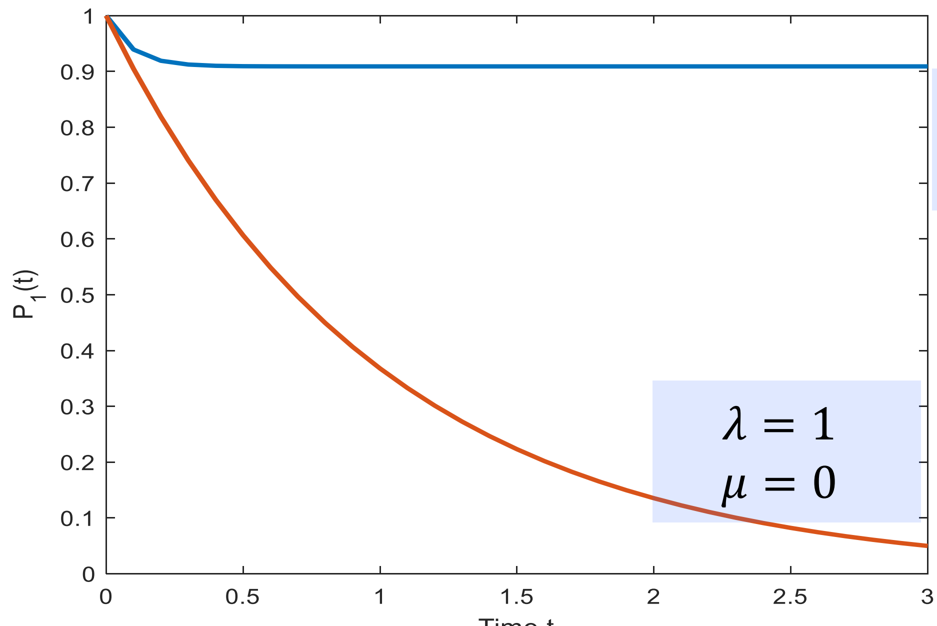


Availability $P_1(t)$

$\lambda = 1$
$\mu = 10$

Survivor function of the component

$R(t) = \Pr(T > t) = e^{-\lambda t}$

Example 4

How to calculate the steady state probabilities?

The Markov models often enter a steady-state after few hours, typically 2-3 times the mean repair time. In this case, it may be of more interest to study steady-state probabilities rather than time-dependent $\mathbf{P}(t)$.

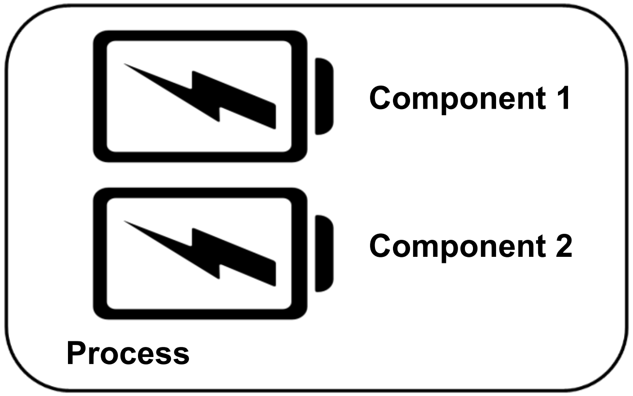
To find the steady-state probabilities for a specific transition model, we have to:

- Step1: Set up the transition matrix (as we did for before)
- Step2: Set up the steady state equations $\mathbf{P} \cdot \mathbb{A} = 0$
- Step3: Solve these equations, bearing in mind that

$$\sum_{j=0}^{r} P_j = 1$$

Example 4

How to calculate the steady state probabilities?



State	Component 1	Component 2
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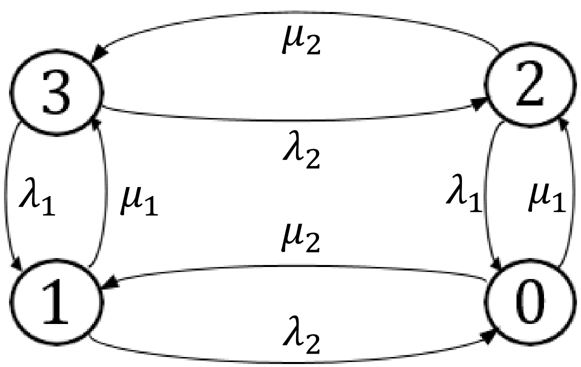
Assume that:

- Each unit has its own repairer. A repair action brings the unit back to functioning
- The units have constant failure rates $\boldsymbol{\lambda_i}$ and constant repair rates $\boldsymbol{\mu_i, i = 1, 2}$
- We only consider the events of single transition

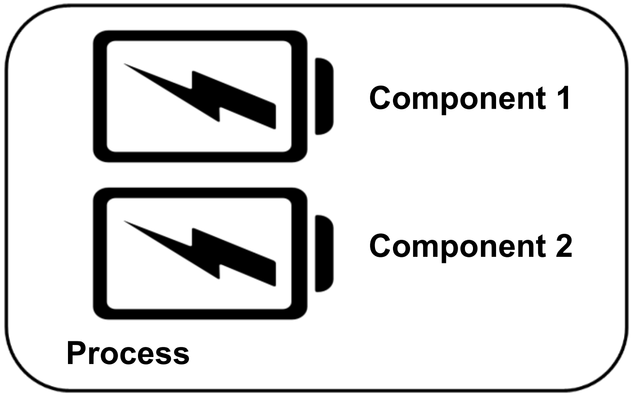


$$\mathbb{A} = \begin{pmatrix} -(\mu_2+\mu_1) & \mu_2 & \mu_1 & 0 \\ \lambda_2 & -(\lambda_2+\mu_1) & 0 & \mu_1 \\ \lambda_1 & 0 & -(\lambda_1+\mu_2) & \mu_2 \\ 0 & \lambda_1 & \lambda_2 & -(\lambda_1+\lambda_2) \end{pmatrix}$$

Markov Process: How to calculate the steady state probabilities? III

Example 4

How to calculate the steady state probabilities?



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Assume that:

- Each unit has its own repairer. A repair action brings the unit back to functioning
- The units have constant failure rates λ_i and constant repair rates $\mu_i, i = 1,2$
- We only consider the events of single transition

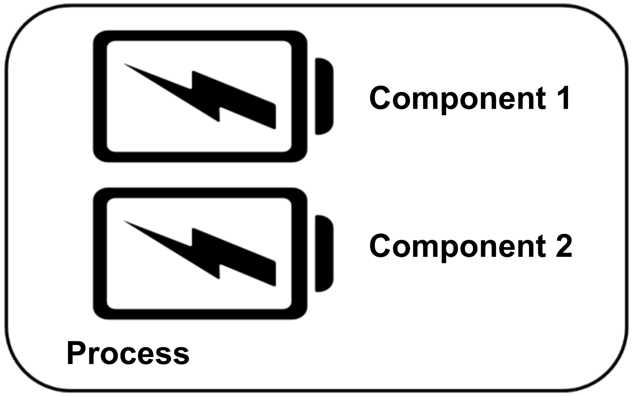
the steady state equations :

$$[P_0, P_1, P_2, P_3]\begin{pmatrix} -(\mu_2+\mu_1) & \mu_2 & \mu_1 & 1 \\ \lambda_2 & -(\lambda_2+\mu_1) & 0 & 1 \\ \lambda_1 & 0 & -(\lambda_1+\mu_2) & 1 \\ 0 & \lambda_1 & \lambda_2 & 1 \end{pmatrix} = [0,0,0,1]$$

Markov Process: How to calculate the steady state probabilities? IV

Example 4

How to calculate the steady state probabilities?



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- We only consider the events of single transition

the steady state equations :

$$-(\mu_2 + \mu_1)P_0 + \lambda_2 P_1 + \lambda_1 P_2 = 0$$

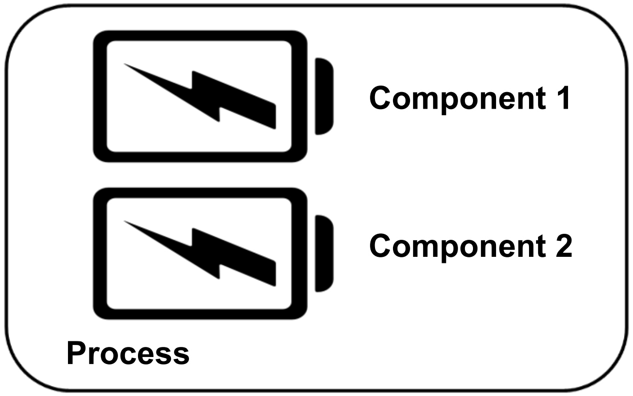
$$\mu_2 P_0 - (\lambda_2 + \mu_1)P_1 + \lambda_1 P_3 = 0$$

$$\mu_1 P_0 - (\lambda_1 + \mu_2)P_2 + \lambda_2 P_3 = 0$$

$$P_0 + P_1 + P_2 + P_3 = 1$$

Example 4

How to calculate the steady state probabilities?



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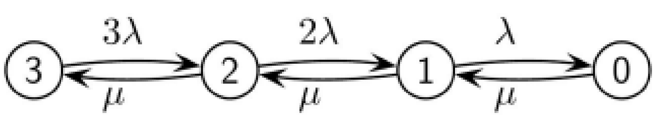
- Each unit has its own repairer. A repair action brings the unit back to functioning
- The units have constant failure rates $\boldsymbol{\lambda_i}$ and constant repair rates $\boldsymbol{\mu_i, i = 1, 2}$
- We only consider the events of single transition

the solution:

$$P_0 = \frac{\lambda_1 \lambda_2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} \qquad P_1 = \frac{\lambda_1 \mu_2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

$$P_2 = \frac{\mu_1 \lambda_2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)} \qquad P_3 = \frac{\mu_1 \mu_2}{(\lambda_1 + \mu_1)(\lambda_2 + \mu_2)}$$

Example 5: Consider a workshop with three critical machines. Each machine has a constant failure rate equal to λ and there is one repair man that can repair failed machines. The rate of repair is μ meaning that the mean repair time is 1/μ. The state variable represent the number of failed machines. Note when the system is in state 3 and all machines are functioning, there are 3 machines that potentially may fail, hence the transition rate from state 3 to state 2 equals 3λ. In state 2 there is only two machines that may fail, hence the transition rate from state 2 to state 1 is 2λ. Since there is only one repair man, all the above-diagonal elements equal the repair rate μ.



$$\mathbf{A} = \begin{bmatrix} -\mu & \mu & 0 & 0 \\ \lambda & -\lambda-\mu & \mu & 0 \\ 0 & 2\lambda & -2\lambda-\mu & \mu \\ 0 & 0 & 3\lambda & -3\lambda \end{bmatrix}$$

Steady state probabilities

State	P_i
3	0.9703
2	2.91E-02
1	5.82E-04
0	5.82E-06

In case of λ=0.001, μ=0.1.

Full production is achieved in 97% of the operating hours. For some 3% one machine is down for corrective maintenance, whereas the probability of two or more failed machines is very low.

Markov Process: Visit frequency

- The visiting frequency, $\boldsymbol{v}$, is one of several system performance that we define for the steady-state situation. The visiting frequency for state $\boldsymbol{j}$, $\boldsymbol{v_j}$, is the unconditional transition rate into state $\boldsymbol{j}$.
- For arrival rate, $\boldsymbol{v}_j^{arr}$, and departure rate, $\boldsymbol{v}_j^{dep}$

$$\boldsymbol{v}_j^{dep} = \alpha_j P_j$$

$$\boldsymbol{v}_j^{arr} = \sum_{k \neq j} a_{kj} P_k$$

- Since in the long run we should fulfil the balance equations stating that the total rate into a state equals the total rate out of that state we get:

$$\boldsymbol{v_j} = \alpha_j P_j = \sum_{k \neq j} a_{kj} P_k$$

Markov Process: Mean time to first passage to a given state I

In some situation we would rather find the mean time to the first time the system enters state ***j***. To solve this problem, we can make state ***j*** an absorbing state. An absorbing state means that we can not leave that state. To make a state absorbing we just remove all arcs out of that state. (by removing the ***j***-th row and the ***j***-th column of transition matrix, denote as $\mathbf{A}_{\mathbf{red}}$)

Laplace transform: A tool to solve differential equations

$$f^*(s) = \mathcal{L}\,[f(t)] = \int_0^\infty e^{-st} f(t)dt$$
$$\mathcal{L}\,[f'(t)] = s\,\mathcal{L}\,[f(t)] - f(0) = sf^*(s) - f(0)$$

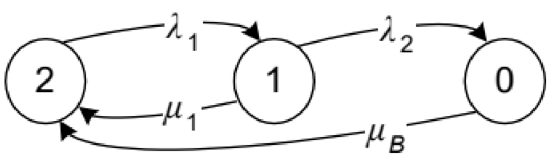
$$R^*(s) = \mathcal{L}\,[R(t)] = \int_0^\infty e^{-st} R(t)dt$$
$$S = 0 \rightarrow R^*(0) = \int_0^\infty e^{-0} R(t)dt = \int_0^\infty R(t)dt = MTTF$$
$$MTTF = R^*(0) = \sum_{i=0, i\neq j}^{r} P_i^*(0)$$

Markov Process: Mean time to first passage to a given state II

Example 6: In a workshop there are two production lines in parallel. Each production line has a critical machine with constant failure rate λ = 0.01 failures per hour. There is one (common) spare machine that can replace a failed machine. We assume that switching time can be ignored. The repair rate of the machines is assumed constant and equal to μ = 0.2 per hour. Only one repair man is available.
NB: Only one repair man is available (which means during the original machine does not works, i.e. during the worker repairs on it, it will no further possible repairment for the stand-by machine if it dose also failed at the same time slot. Thus, no link from state 0 to 1. In addition, repair rate from state 0 to 2, is used to show an example how to calculate visiting frequency of state 0. The reality situation/assumption might be different.)

- Draw the Markov diagram with the transition rates
- Visiting frequencies
- Mean time to first system failure

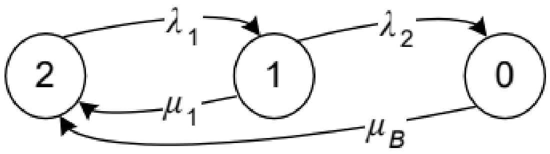
State	Explanation
2	Active pump is running
1	Active pump failed, stand-by pump running
0	Both pumps failed



Rate	Explanation
λ_1	failure rate of the active pump
λ_2	failure rate of the standby pump (while running, $\lambda_2 = 0$ in standby position)
μ_1	repair rate of the active pump
μ_B	repair rate when both pumps are in a fault state

Markov Process: Mean time to first passage to a given state III

State	Explanation
2	Active pump is running
1	Active pump failed, stand-by pump running
0	Both pumps failed



Rate	Explanation
λ_1	failure rate of the active pump
λ_2	failure rate of the standby pump (while running, $\lambda_2 = 0$ in standby position)
μ_1	repair rate of the active pump
μ_B	repair rate when both pumps are in a fault state

$$\mathbf{A} = \begin{bmatrix} a_{00} & a_{01} & a_{02} \\ a_{10} & a_{11} & a_{12} \\ a_{20} & a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} -\mu_B & 0 & \mu_B \\ \lambda_2 & -\lambda_2-\mu_1 & \mu_1 \\ 0 & \lambda_1 & -\lambda_1 \end{bmatrix}$$

$$[P_0, P_1, P_2] \cdot \begin{bmatrix} -\mu_B & 0 & 1 \\ \lambda_2 & -\lambda_2-\mu_1 & 1 \\ 0 & \lambda_1 & 1 \end{bmatrix} = [0,0,1]$$

$$P_2 = \frac{\mu_B(\lambda_2+\mu_1)}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

$$P_1 = \frac{\mu_B\lambda_1}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

$$P_0 = \frac{\lambda_1\lambda_2}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

$$\nu_0 = -P_0 a_{00} = -P_0(-\mu_B) = \frac{\mu_B\lambda_1\lambda_2}{\lambda_1(\lambda_2+\mu_B)+\mu_B(\lambda_2+\mu_1)}$$

$$[P_1^*, P_2^*] \cdot \begin{bmatrix} -\lambda_2-\mu_1 & \mu_1 \\ \lambda_1 & -\lambda_1 \end{bmatrix} = [0,-1]$$

$$\mathrm{MTTF}_S = P_1^* + P_2^* = (\lambda_1+\lambda_2+\mu_1)/(\lambda_1\lambda_2)$$