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STOCHASTIC PROGRAMMING

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Motivation

In the standard linear programming problem we assumed all quantities to be know in advance. That is, the LP problem formulation was given by:

$$\text{Maximize: } Z = \mathbf{cx}$$

$$\begin{aligned} \text{Subject to: } \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq 0 \\ \mathbf{b} &\geq 0 \end{aligned}$$

Notation

- $\mathbf{x}$ is a *decision vector* which is controlled by the decision maker
- $\mathbf{c}$ is the *profit/cost vector*
- $\mathbf{A}$ is the *coefficient matrix*
- $\mathbf{b}$ is the *requirement vector*

Up to now we have assumed that $\mathbf{c}$, $\mathbf{A}$ and $\mathbf{b}$ are all known at the time of the decision to be made. In real life, we often have to make decisions when some of these quantities are not known. Typically the requirement vector represents resource constraints which are only or partly known at a later stage in the decision process

- Stochastic programming deals with such uncertainties

Motivation

- Fish processing factory
- Today we need to plan tomorrows production
- Optimal production depends on the catch, which we do not know before tomorrow
- Possible catch:
  - High quality can go into both fresh fillet and frozen fillet
  - Low quality can only go into frozen fillet
- Today we have to decide upon x = number of workers to come to work tomorrow
- Tomorrow we have to decide upon $\mathbf{y}$ = production plan for tomorrow, when the catch is known

Notation

- Today’s decision **x** is often referred to as the first stage decision, or the here and now decision
- Tomorrow’s decision **y** is referred to as the second stage decision, and is made when the uncertainty is revealed
- We assume that we today can describe the uncertainty regarding tomorrow’s situation by some stochastic vector **U**, here representing the catch

Objective function

- The objective function is now split into two objectives function, Z_1 and Z_2 The maximization of $Z_1 = \mathbf{cx}$ subject to some $\mathbf{Ax} = \mathbf{b}$ is denoted the *first stage* optimization
- Given that we know the value of the stochastic vector, say $\mathbf{U} = \mathbf{u}$ there is a second stage optimization problem defined by:

$$\text{Maximize: } Z_2 = \mathbf{q}(\mathbf{u})\mathbf{y}$$

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$$\text{Maximize: } Z_2 = \mathbf{q}(\mathbf{u})\mathbf{y}$$

$$\begin{aligned} \text{Subject to: } \mathbf{B}(\mathbf{u})\mathbf{x} + \mathbf{C}(\mathbf{u})\mathbf{y} &= \mathbf{d}(\mathbf{u}) \\ \mathbf{y} &\geq 0 \\ \mathbf{d}(\mathbf{u}) &\geq 0 \end{aligned}$$

Objective function, cont.

- Since the decision to make tomorrow is influenced by the decision we make today through the constraints $\mathbf{B}(\mathbf{u})\mathbf{x}$ the decisions the two days are *interconnected*
- In the first stage we need to take into account what will be the result of the second stage decision but this again depends on the unknown vector $\mathbf{U}$
- The two stage stochastic programming formulation is now:

$$\text{Maximize: } Z_{1,2} = \mathbf{cx} + \mathrm{E}_{\mathbf{U}}\left[Q(\mathbf{x}, \mathbf{U})\right]$$

$$\begin{aligned} \text{Subject to: } \mathbf{Ax} &= \mathbf{b} \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{b} &\geq \mathbf{0} \end{aligned}$$

Objective function, cont.

In $\mathrm{E}_{\mathbf{U}}\left[Q(\mathbf{x},\mathbf{U})\right]$ the expectation is taken with respect to the stochastic vector $\mathbf{U}$ and $Q(\mathbf{x},\mathbf{u})$ is the solution to:

$$\text{Maximize: } Z_2 = \mathbf{q}(\mathbf{u})\mathbf{y}$$

$$\begin{aligned} \text{Subject to: } \mathbf{B}(\mathbf{u})\mathbf{x} + \mathbf{C}(\mathbf{u})\mathbf{y} &= \mathbf{d}(\mathbf{u}) \\ \mathbf{y} &\geq \mathbf{0} \\ \mathbf{d}(\mathbf{u}) &\geq \mathbf{0} \end{aligned}$$

- Both $\mathbf{q}(\mathbf{u})$ and $\mathbf{d}(\mathbf{u})$ depend on the value, u, i.e., the value the stochastic vector $\mathbf{U}$ will take
- $\mathbf{B}(\mathbf{u})\mathbf{x} + \mathbf{C}(\mathbf{u})\mathbf{y}$ is built up by two terms: $\mathbf{B}(\mathbf{u})$ relates to the first stage decision vector $\mathbf{x}$ and $\mathbf{C}(\mathbf{u})$ relates to the second stage decision vector $\mathbf{y}$
- Both matrices depend on the value the stochastic vector $\mathbf{U}$ will take

Discretization

- To solve the two-stage stochastic problem numerically it is common to perform a discretization where the stochastic vector $\mathbf{U}$ can only take a finite number of possible realizations
- These realizations are often denoted scenarios, say $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k$, with corresponding probabilities $p_i = \Pr(\mathbf{U} = \mathbf{u}_i)$
- The expectation in the second-stage problem is then given by

$$\mathrm{E}_{\mathbf{U}}\left[Q(\mathbf{x}, \mathbf{U})\right] = \sum_{i=1}^{k} p_i Q(\mathbf{x}, \mathbf{u}_i)$$

The deterministic equivalent problem now reads

$$\text{Maximize: } Z_{1,2} = \mathbf{cx} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\text{Subject to: } \mathbf{Ax} = \mathbf{b}$$

$$\mathbf{B}(\mathbf{u}_i)\mathbf{x} + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i = \mathbf{d}(\mathbf{u}_i),\ i = 1, 2, \ldots, k$$

The deterministic equivalent problem now reads

$$\text{Maximize: } Z_{1,2} = \mathbf{cx} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\begin{aligned} \text{Subject to: } \mathbf{Ax} &= \mathbf{b} \\ \mathbf{B}(\mathbf{u}_i)\mathbf{x} + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i &= \mathbf{d}(\mathbf{u}_i),\, i = 1, 2, \ldots, k \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{b} &\geq \mathbf{0} \\ \mathbf{y}_i &\geq \mathbf{0} \\ \mathbf{d}(\mathbf{u}_i) &\geq \mathbf{0},\, i = 1, 2, \ldots, k \end{aligned}$$

For the scenarios $i = 1, 2, \ldots, k$ we have to specify $\mathbf{B}(\mathbf{u}_i), \mathbf{C}(\mathbf{u}_i)$ and $\mathbf{d}(\mathbf{u}_i)$ one by one based on the understanding of the problem at hand

Typical structure

First stage	Scenario 1	Scenario 2	Scenario 3		
$c_1, c_2, c_3 \dots$	$p_1 q_{11}, p_1 q_{12}, \dots$	$p_2 q_{12}, p_2 q_{22}, \dots$	$p_3 q_{13}, p_3 q_{23}, \dots$	$\dots$	
$x_1, x, x_3 \dots$	$y_{11}, y_{21}, \dots$	$y_{12}, y_{22}, \dots$	$y_{13}, y_{23}, \dots$	$\dots$	
$\mathbf{B}_1$	$\mathbf{C}_1$	$\mathbf{0}$	$\mathbf{0}$	$=$	$\mathbf{d}_1$
$\mathbf{B}_2$	$\mathbf{0}$	$\mathbf{C}_2$	$\mathbf{0}$	$=$	$\mathbf{d}_2$
$\mathbf{B}_3$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{C}_3$	$=$	$\mathbf{d}_3$
$\vdots$					

- The second row above the horizontal line corresponds to the profit parameters
- The first row under the horizontal line corresponds to the first scenario, i.e., $\mathbf{B}(\mathbf{u}_1)\mathbf{x} + \mathbf{C}(\mathbf{u}_1)\mathbf{y}_1 = \mathbf{d}(\mathbf{u}_1)$
- The second row corresponds to $\mathbf{B}(\mathbf{u}_2)\mathbf{x} + \mathbf{C}(\mathbf{u}_2)\mathbf{y}_2 = \mathbf{d}(\mathbf{u}_2)$
- etc.

Example: Fish processing facility

We consider a fish processing facility where tomorrow's supply is uncertain. A discretization into 3 different supply scenarios is performed:

Scenario, i	p_i	High quality	Low quality
1	0.15	16 000	12 000
2	0.5	8 000	6 000
3	0.35	6 000	3 000

Parameters

Parameter	Value	Description
ρ_{Fi}	0.6	Utilization ratio fresh fillet
ρ_{Fz}	0.7	Utilization ratio frozen fillet
p_{Fi}	80	Selling price, fresh fillet (NOKs/kg)
p_{Fz}	30	Selling price, frozen fillet (NOKs/kg)
μ_{Fi}	70	Processing capacity, kg fresh fillet /man hour
μ_{Fz}	300	Processing capacity, kg frozen fillet /man hour
c_{MHR}	500	Man hour cost (NOKs)

The following decision variables are introduced

- x_1 = Number of man-hours allocated
- $y_{1,i}$ = Number of kg processed for fresh fish fillet, scenario i
- $y_{2,i}$ = Number of kg processed for frozen fish fillet, scenario i

Profit coefficients corresponding to the decision variables:

- x_1: $-c_{\mathrm{MHR}}$
- $y_{1,i}$: $p_i \rho_{\mathrm{Fi}} p_{\mathrm{Fi}}$
- $y_{2,i}$: $p_i \rho_{\mathrm{Fz}} p_{\mathrm{Fz}}$

Constraints

In the constraint matrix for the deterministic equivalent problem we have for each scenario to make sure that

- $y_{1,i}$ not exceeds supply of high quality fish
- $y_{1,i} + y_{2,i}$ not exceeds total supply of fish
- $y_{1,i}/\mu_{\mathrm{Fi}} + y_{2,i}/\mu_{\mathrm{Fz}} \leq x_1$

Excel model

Variables	x_1	y_{11}	y_{21}	y_{12}	y_{22}	y_{13}	y_{23}
c_j-Values	-500	7.2	3.15	24	10.5	14.4	6.3
Solution	134.29	3739	24261	8000	6000	6000	3000
Z =	396 501						

Constraints	a_{i1}	a_{i2}	a_{i3}	a_{i4}	a_{i5}	a_{i6}	a_{i7}	b_i	$\Sigma a_{ij} x_j - b_i$	
		1						16000	-12260.87	Constraints on fresh fish fillet, scenario 1
		1	1					28000	0	Constraints on frosen fish fillet, scenario 1
				1				8000	0	Constraints on fresh fish fillet, scenario 2
				1	1			14000	0	Constraints on frosen fish fillet, scenario 2
						1		6000	0	Constraints on fresh fish fillet, scenario 3
						1	1	9000	0	Constraints on frosen fish fillet, scenario 3
	-1	0.014	0.003						1.037E-12	Constraints on work-force, scenario 1
	-1			0.014	0.003				6.253E-13	Constraints on work-force, scenario 2
	-1					0.014	0.003		-38.57143	Constraints on work-force, scenario 3

▸ Show in Excel ▸ Show in Python

The Value of the Stochastic Solution - VSS

- The Value of the Stochastic Solution (VSS) is introduced to measure the benefit of solving the stochastic problem rather than treating the problem as a deterministic problem
- With the “deterministic problem” we mean treating the problem as a deterministic problem where the stochastic variables are replaced by their *expected values*, i.e., ignoring the uncertainties affecting the second stage decision
- This is referred to as an expected value (EV) approach
- In the EV approach we solve the two stage problem by anticipating the value of **U** by using $\bar{\mathbf{u}} = \mathrm{E}(\mathbf{U})$

The EV solution follows from:

$$\text{Maximize: } Z_{\mathrm{E}V} = \mathbf{cx} + \mathbf{q}(\bar{\mathbf{u}})\mathbf{y}$$

$$\begin{aligned} \text{Subject to: } \mathbf{Ax} &= \mathbf{b} \\ \mathbf{B}(\bar{\mathbf{u}})\mathbf{x} + \mathbf{C}(\bar{\mathbf{u}})\mathbf{y} &= \mathbf{d}(\bar{\mathbf{u}}) \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{b} &\geq \mathbf{0} \\ \mathbf{y} &\geq \mathbf{0} \\ \mathbf{d}(\bar{\mathbf{u}}) &\geq \mathbf{0} \end{aligned}$$

The expected value of the expected value solution

The expected value of the expected value solution is now the value of the following optimization problem ($\mathbf{cx}_{\mathrm{EV}}$ fixed!):

$$\text{Maximize: } Z_{\mathrm{EEV}} = \mathbf{cx}_{\mathrm{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\begin{aligned} \text{Subject to: } \mathbf{B}(\mathbf{u}_i)\mathbf{x}_{\mathrm{EV}} + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i &= \mathbf{d}(\mathbf{u}_i), i = 1, 2, \ldots, k \\ \mathbf{y}_i &\geq \mathbf{0} \\ \mathbf{d}(\mathbf{u}_i) &\geq \mathbf{0}, i = 1, 2, \ldots, k \end{aligned}$$

Here the solution, $\mathbf{y_i}$, for each scenario *i* could be found individually

The value of the stochastic solution

- Let Z^*_{EEV} be the value of the optimization problem:

$$\text{Maximize: } Z_{\mathrm{EEV}} = \mathbf{c}\mathbf{x}_{\mathrm{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

- Let $Z^*_{1,2}$ be value of the optimization problem:

$$\text{Maximize: } Z_{1,2} = \mathbf{c}\mathbf{x} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

- The difference between these two is denoted the value of the stochastic solution:

$$\mathrm{VSS} = Z^*_{1,2} - Z^*_{\mathrm{EEV}}$$

Expected value of perfect information

- In decision theory, the expected value of perfect information (EVPI) is the price that one would be willing to pay in order to gain access to perfect information
- In our context this means the difference between the expected value of the stochastic solution and expected value of the solution if we knew **u** at the time of the first stage decision
- For the latter case we often say that this correspond to a wait and see situation were we imagine that we can wait and observe **u** before we need to decide upon **x**

Expected value of perfect information, cont.

The optimization problem for the wait and see situation given that we observe $\mathbf{u} = \mathbf{u}_i$ is given by:

$$\text{Maximize: } Z_i = \mathbf{c}\mathbf{x}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\begin{aligned}
\text{Subject to: } \mathbf{A}\mathbf{x}_i &= \mathbf{b} \\
\mathbf{B}(\mathbf{u}_i)\mathbf{x}_i + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i &= \mathbf{d}(\mathbf{u}_i) \\
\mathbf{x} &\geq \mathbf{0} \\
\mathbf{b} &\geq \mathbf{0} \\
\mathbf{y}_i &\geq \mathbf{0} \\
\mathbf{d}(\mathbf{u}_i) &\geq \mathbf{0}
\end{aligned}$$

Expected value of perfect information, cont.

- Let Z_i^* be the result of: Maximize: $Z_i = \mathbf{cx}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$
- The expected profit when averaging over all scenarios is given by $\sum_i p_i Z_i^*$

Expected value of perfect information, cont.

- Let Z_i^* be the result of: Maximize: $Z_i = \mathbf{c}\mathbf{x}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$
- The expected profit when averaging over all scenarios is given by $\sum_i p_i Z_i^*$
- The expected value of perfect information is thus:

$$\text{EVPI} = \sum_{i=1}^{k} p_i Z_i^* - Z_{1,2}^*$$

Example 2

The example is motivated from Exam 2019

- PDF
- Excel

Thank you for your attention

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