# Problem 1-2

We are going to prove that

$$B(x_0,\mu,\sigma) = \int_{x_0}^{\infty} (x - x_0) f_X(x)\, dx$$
$$= (\mu - x_0)\left[1 - \Phi\left(\frac{x_0-\mu}{\sigma}\right)\right] + \sigma\phi\left(\frac{x_0-\mu}{\sigma}\right)$$

where $\Phi()$ and $\phi()$ are the cumulative distribution function and probability density function for the standard normal distribution respectively.

We utilize the result for the expected value of a truncated normal distributed variable:

$$\int_{-\infty}^{a} x f(x) dx = \mu\Phi\left(\frac{a-\mu}{\sigma}\right) - \sigma\phi\left(\frac{a-\mu}{\sigma}\right)$$

Thus, we have

$$B(x_0,\mu,\sigma) = \int_{x_0}^{\infty} (x - x_0) f_X(x)\, dx = \int_{x_0}^{\infty} x f_X(x)\, dx - x_0 \int_{x_0}^{\infty} f_X(t)\, dx$$
$$= \mu - \int_{-\infty}^{x_0} x f_X(x)\, dx - x_0\left[1 - \Phi\left(\frac{x_0-\mu}{\sigma}\right)\right]$$
$$= \mu - \mu\Phi\left(\frac{x_0-\mu}{\sigma}\right) + \sigma\phi\left(\frac{x_0-\mu}{\sigma}\right) - x_0\left[1 - \Phi\left(\frac{x_0-\mu}{\sigma}\right)\right]$$
$$= (\mu - x_0)\left[1 - \Phi\left(\frac{x_0-\mu}{\sigma}\right)\right] + \sigma\phi\left(\frac{x_0-\mu}{\sigma}\right)$$

Where we use that the expected value in the normal distribution is

$$\mu = \int_{-\infty}^{\infty} x f_X(x)\, dx = \int_{-\infty}^{a} x f_X(x)\, dx + \int_{a}^{\infty} x f_X(x)\, dx$$