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THE FARMERS PROBLEM

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Situation

- Farmer Tom can grow wheat, corn, and sugar beets on his 500 acres
- Yearly demand for feeding the cattle: 200 tons of wheat and 240 tons of corn
- Production in excess of these amounts can be sold for $170/ton (wheat) and $150/ton (corn)
- Any shortfall must be bought from the wholesaler at a cost of $238/ton (wheat) and $210/ton (corn)
- Sugar beets can also be grown and sold for $36/ton for the first 6 000 tons, and excess of 6 000 tons can only be sold at $10/ton

Situation, cont

Planting costs:

- Wheat: $150/acre
- Corn: $230/acre
- Sugar beets: $260/acre

Yield in a normal year:

- Wheat: 2.5 tons/acre
- Corn: 3 tons/acre
- Sugar beets: 20 tons/acre

If case of bad weather the yield is 80% and if the weather is good the yield is 120% of the normal yield. Equal probabilities to the 3 scenarios.

Stochastic programming

- Tom needs to sow in the spring before he knows the weather for the coming season
- Hence, the first stage decision is what to sow, and the second stage decision is what to do after the harvesting.
- First stage decision variables:
  - x_1 = Acres allocated to wheat
  - x_2 = Acres allocated to corn
  - x_3 = Acres allocated to sugar beats

Second stage decisions

Second stage decision variables:

- $w_{1,i}$ Tons of wheat sold, scenario i
- $w_{2,i}$ Tons of corn sold, scenario i
- $w_{3,i}$ Tons of sugar beats sold at favourable price, scenario i
- $w_{4,i}$ Tons of sugar beats sold at lower price, scenario i
- $y_{1,i}$ Tons of wheat purchased, scenario i
- $y_{2,i}$ Tons of corn purchased, scenario i

Scenario 1/2/3 = Good/Normal/Bad weather respectively

Parameters

Prices:

- $s_1 = 170$ = Sales price, wheat
- $s_2 = 150$ = Sales price, corn
- $s_3 = 36$ = Sales price, sugar beats favourable
- $s_4 = 10$ = Sales price, sugar beats low

Costs:

- $c_1 = 150$ = Planting cost, wheat/acre
- $c_2 = 230$ = Planting cost, corn/acre
- $c_3 = 260$ = Planting cost, sugar beats/acre
- $b_1 = 238$ = Cost of buying from the wholesaler, wheat
- $b_2 = 210$ = Cost of buying from the wholesaler, corn

Final parameters

- $r_1 = 2.5$ = Yield per acre, wheat
- $r_2 = 3$ = Yield per acre, corn
- $r_3 = 20$ = Yield per acre, sugar beets
- $d_1 = 200$ = Demand/requirements for Tom's cattle, corn
- $d_2 = 240$ = Demand/requirements for Tom's cattle, wheat
- $p_1 = 1/3$ = Probability, good weather
- $p_2 = 1/3$ = Probability, normal weather
- $p_3 = 1/3$ = Probability, bad weather

Recall: The deterministic equivalent problem

Maximize: $Z_{1,2} = \mathbf{cx} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$

Subject to: $\mathbf{Ax} = \mathbf{b}$

$$\mathbf{B}(\mathbf{u}_i)\mathbf{x} + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i = \mathbf{d}(\mathbf{u}_i),\ i = 1, 2, \ldots, k$$

In our case: Objective function

$$\text{Maximize: } Z = -\sum_{j=1}^{3} c_i x_i + \sum_{i=1}^{3} p_i \left(\sum_{j=1}^{4} s_i w_{j,i} - \sum_{j=1}^{2} b_i y_{j,i} \right)$$

Constraints, i.e., Subject to:

Use of land:

$$\sum_{j=1}^{3} x_j \leq 500$$

Sugar beats, scenario *i*:

$$w_3 + w_4 \leq r_3 u_i x_3$$
$$w_3 \leq 6000$$

where u_i = yield factor (1.2, 1 and 0.8) depends on the scenario

Constraints, i.e., Subject to:

Wheat:

$$x_1 r_1 u_i - w_{1,i} + y_{1,i} \geq d_1$$

Corn:

$$x_2 r_2 u_i - w_{2,i} + y_{2,i} \geq d_2$$

▸ Solution in Python

Solution

Variable	Total	Scenario 1	Scenario 2	Scenario 3
x_1		170	170	170
x_2		80	80	80
x_3		250	250	250
w_{1i}		310	225	140
y_{1i}		0	0	0
w_{2i}		48	0	0
y_{2i}		0	0	48
w_{3i}		6 000	5 000	4 000
w_{4i}		0	0	0
Z	108 390	167 000	109 350	48 820

About the Python implementation

- In Python we write a function: `spSolve(p)`
- where `p` is the probability vector for each scenario
- This will enable us to force the model to treat a specific scenario only, e.g., `p = [1,0,0]`
- In particular using `p = [0,1,0]` gives the expected value solution
- Later on, we also add an `x`-vector that can “lock” the first stage decision

The EV solution follows from:

$$\text{Maximize: } Z_{\mathrm{E}V} = \mathbf{cx} + \mathbf{q}(\overline{\mathbf{u}})\mathbf{y}$$

$$\begin{aligned} \text{Subject to: } \mathbf{Ax} &= \mathbf{b} \\ \mathbf{B}(\overline{\mathbf{u}})\mathbf{x} + \mathbf{C}(\overline{\mathbf{u}})\mathbf{y} &= \mathbf{d}(\overline{\mathbf{u}}) \\ \mathbf{x} &\geq \mathbf{0} \\ \mathbf{b} &\geq \mathbf{0} \\ \mathbf{y} &\geq \mathbf{0} \\ \mathbf{d}(\overline{\mathbf{u}}) &\geq \mathbf{0} \end{aligned}$$

The EV solution - Python considerations

- We are to: Maximize: $Z_{EV} = \mathbf{cx} + \mathbf{q}(\bar{\mathbf{u}})\mathbf{y}$
- In Python we can easily solve this by putting all probability mass to scenario 2, i.e., the "average scenario", i.e., we call the function `spSolve([0,1,0])`
- This will give us both the first and second stage decision variable values
- The second stage values are of not interest wrt. the EV solution, but
  - In the next step we need to "lock" the first stage variables
  - Let $\mathbf{x}_{EV}$ be a vector defining the "locked" first stage variables.

Python considerations, Z_{EEV}

- The Z_{EEV} value is obtained by running the `spSolve()` function with the original **p**-vector, and locking the first stage variables, i.e.,
- `spSolve(p,x_EV)`
- where `x_EV` is the expected value solution for the first stage variable

The expected value of the expected value solution

The expected value of the expected value solution is now the value of the following optimization problem:

$$\text{Maximize: } Z_{\mathrm{EEV}} = \mathbf{c}\mathbf{x}_{\mathrm{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\text{Subject to: } \mathbf{B}(\mathbf{u}_i)\mathbf{x}_{\mathrm{EV}} + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i = \mathbf{d}(\mathbf{u}_i),\ i = 1, 2, \ldots, k$$

$$\mathbf{y}_i \geq \mathbf{0}$$

$$\mathbf{d}(\mathbf{u}_i) \geq \mathbf{0},\ i = 1, 2, \ldots, k$$

Here the solution, $\mathbf{y}_\mathbf{i}$, for each scenario *i* could be found individually

The expected value of the expected value solution - VBA considerations

- We are to maximize $Z_{\text{EEV}} = \mathbf{cx}_{\text{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$
- We now have to options:
  (a) Maximize $\mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$ for each scenario individually, find the weighted average $\sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$ and add $\mathbf{cx}_{\text{EV}}$
  (b) Maximize wrt all $\mathbf{y}_i$-vectors simultaneously, i.e., maximize
  $$Z_{\text{EEV}} = \mathbf{cx}_{\text{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$
- Option (a) is much faster since the problem is reduced, but more tedious
- Pptions (b) is implemented by `spSolve(p,x_EV)`

Results, expected value solution

Variable	Total	Scenario 1	Scenario 2	Scenario 3
x_1		120	120	120
x_2		80	80	80
x_3		300	300	300
w_{1i}		160	100	40
y_{1i}		0	0	0
w_{2i}		48	0	0
y_{2i}		0	0	48
w_{3i}		6 000	6 000	4 800
w_{4i}		1 200	0	0
Z	107 240	148 000	118 600	55 120

The value of the stochastic solution

- Let the value of the optimization problem: $Z_{\mathrm{EEV}} = \mathbf{c}\mathbf{x}_{\mathrm{EV}} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$ be denoted Z^*_{EEV}
- Let the value of the optimization problem:

  Maximize: $Z_{1,2} = \mathbf{c}\mathbf{x} + \sum_{i=1}^{k} p_i \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$ be denoted $Z^*_{1,2}$
- The difference between these two is denoted the value of the stochastic solution:

$$\mathrm{VSS} = Z^*_{1,2} - Z^*_{\mathrm{EEV}} = 108\ 390 - 107\ 240 = 1\ 150$$

Discussion

- The highest relative profit is achieved by growing sugar beats
- However, this is true only up to 6 000 ton
- In the EV solution, we therefore maximize to achieve exactly 6 000 ton of sugar beats given a normal season
- In case of a good season, we then exceed this limit an the *value* of a good season is limited since we have to sell at a low price
- In the stochastic solution, on the other hand, we grow less sugar beats and more wheat
- We observe that for the stochastic solution, we grow so much sugar beats that we exactly reach 6 000 ton for the good season, and hence never have to sell at the low price

Expected value of perfect information

The optimization problem for the "wait and see" situation given that we observe $\mathbf{u} = \mathbf{u}_i$ is given by:

$$\text{Maximize: } Z_i = \mathbf{c}\mathbf{x}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$$

$$\text{Subject to: } \mathbf{A}\mathbf{x}_i = \mathbf{b}$$

$$\mathbf{B}(\mathbf{u}_i)\mathbf{x}_i + \mathbf{C}(\mathbf{u}_i)\mathbf{y}_i = \mathbf{d}(\mathbf{u}_i)$$

$$\mathbf{x} \geq \mathbf{0}$$

$$\mathbf{b} \geq \mathbf{0}$$

$$\mathbf{y}_i \geq \mathbf{0}$$

$$\mathbf{d}(\mathbf{u}_i) \geq \mathbf{0}$$

Expected value of perfect information, cont.

- Let Z_i^* be the result of: Maximize: $Z_i = \mathbf{c}\mathbf{x}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$
- The expected profit when averaging over all scenarios is given by $\sum_i p_i Z_i^*$

Expected value of perfect information, cont.

- Let Z_i^* be the result of: Maximize: $Z_i = \mathbf{c}\mathbf{x}_i + \mathbf{q}(\mathbf{u}_i)\mathbf{y}_i$
- The expected profit when averaging over all scenarios is given by $\sum_i p_i Z_i^*$
- The expected value of perfect information is thus:

$$\text{EVPI} = \sum_{i=1}^{k} p_i Z_i^* - Z_{1,2}^* \approx 115\,406 - 108\,390 = 7\,015$$

Intermediate results, wait and see solutions

Variable	Total	Scenario 1	Scenario 2	Scenario 3
x_1		183.3	120	100
x_2		66.7	80	25
x_3		250	300	375
w_{1i}		350	100	0
y_{1i}		0	0	0
w_{2i}		0	0	0
y_{2i}		0	0	180
w_{3i}		6 000	6 000	6 000
w_{4i}		0	0	0
Z	115 406	167 667	118 600	59 950

Thank you for your attention

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