Project Risk Analysis

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August, 18th 2013

Contents

Chapter 1

Introduction

1.1 About this compendium

This course compendium is to be used in various cources at NTNU. The focus will be on the following topics:

- Risk identification
- Risk structuring
- Risk modelling in the light of a time schedule and a cost model
- Risk follow up

We will also discuss elements related to decision analysis where risk is involved, and use of life cycle cost and life cycle profit models.

The course compendium comprises a large number of exercises, and it is recommended to do most of the exercises in order to get a good understanding of the topics and methods described. A separate MS Excel program, pRisk.xls has been developed in order to assist numerical calculations and to conduct Monte Carlo simulation.

1.2 Definitions

Aleatory uncertainty
Variation of quantities in a population. We sometimes use the word variability rather than aleatory uncertainty.

Epistemic uncertainty
Lack of knowledge about the “world”, and observable quantities in particular.

Dependency
The relation between the sequence of the activities in a project.

Observable quantity
A quantity expressing a state of the "world", i.e. a quantity of the physical reality or nature, that is unknown at the time of the analysis but will, if the system being analysed is actually implemented, take some value in the future, and possibly become known.

Parameter
We use the term parameter in two ways in this report. The main use of a parameter is that it is a quantity that is a part of the risk analysis models, and for which we assign numerical values. The more academic definition of a parameter used in a probability statement about an *observable quantity*, X, is that a parameter is a construct where the value of the parameter is the limiting value where we are not able to saturate our understanding about the observable quantity X whatsoever new information we could get hold of.

Parameter estimate
The numeric value we assess to a parameter.

Probability
A measure of uncertainty of an event.

Risk
Risk is uncertainty regarding occurence and severity of future events. In project risk management focus is on undesired events, whereas economists also include opportunities as part of risk. To quantify risk three elements are introduced: $< e, p, \mathbf{S} >$. p is used as a probability measure of the occurrence of an event, say e. $\mathbf{S}$ represents the severity of the event. Note that $\mathbf{S}$ is a multidimensional random quantity, covering several dimensions like personnel safety, environmental impacts, material damages, project dealys, extra costs, etc. Since there is more than one event to treat, i is used as an index to run through all relevant events. An operational definition of risk is thus the set of all relevant triplets: $R = \{< e_i, p_i, \mathbf{S}_i >\}$.

Risk picture
A set of undesired events, the causes and factors that may contribute to the event, the possible consequences of the event with corresponding influencing factors, and uncertainties related to all these issues.

Risk acceptance
A decision to accept a risk.

Risk acceptance criterion
A reference by which risk is assessed to be acceptable or unacceptable.

Schedule
A plan which specifies the start and finalisation point of times for the activities in a project.

Stochastic dependency
Two or more stochastic variables are (stochastically) dependent if the expectation of one stochastic variable depends on the value of one or more of the other stochastic variables.

Stochastic variable
A stochastic variable, or random quantity, is a quantity for which we do not know the value it will take. However, we could state statistical properties of the variable or make probability statement about the value of the quantity.

Uncertainty
Lack of knowledge about the performance of a system, and observable quantities in particular.

Chapter 2

Risk Management

Generally, risk management is defined (IEC 60300-3-9) as a "systematic application of management policies, procedures and practices to the tasks of analyzing, evaluating and controlling risk". It will comprise (IEC definitions in parentheses):

- Risk assessment, i.e.
  - Risk analysis ("Systematic use of available information to identify hazards and to estimate the risk to individuals or populations, property or the environment"), and
  - Risk evaluation ("Process in which judgments are made on the tolerability of the risk on the basis of risk analysis and taking into account factors such as socio-economic and environmental aspects")
- Risk reduction/control (Decision making, implementation and risk monitoring).

There exists no common definition of risk, but for instance IEC 60300-3-9 [13] defines risk as a "combination of the frequency, or probability, of occurrence and the consequence of a specified hazardous events". Most definitions comprise the elements of probabilities and consequences. However, some as Klinke and Renn (2001) [15] suggest a very wide definition, stating: "Risk refers to the possibility that human actions or events lead to consequences that affect aspects of what humans value". So the total risk comprises the possibility of a number ("all") unwanted/hazardous events. It is part of the risk analysis to delimit which hazards to include. Further, risk usually refers to threats in the future, involving a (high) degree of uncertainty. In this presentation risk is defined as uncertainty regarding occurence and severity of future events. To make an operational definition of risk three elements are introduced: $< e, p, \mathbf{S} >$. p is used as a probability measure of the occurrence of an event, say e. S represents the severity of the event. Note that $\mathbf{S}$ is a multidimensional random quantity, covering several dimensions like personnel safety, environmental impacts, material damages, project dealys, extra costs, etc. Since there is more than one event to treat, i is used as an index to run through all relevant events. An operational definition of risk is thus the set of all relevant triplets: $R = \{< e_i, p_i, \mathbf{S}_i >\}$.

In the following we will present the basic elements of risk management as it is proposed to be an integral part of project management.

2.1 Project objectives and criteria

In classical risk analysis of industrial systems the use of so-called risk acceptance criteria has played a central role in the last two or tree decades. Basically use of risk acceptance criteria means that some severe consequences are defined, e.g. accident with fatalities. Then we try to set an upper limit for the probability of these consequences that could be accepted, i.e. we could not accept higher probabilities in any situations. Further these probabilities could only be accepted if risk reduction is not possible, or the cost of risk reduction is very high.

In recent years it has been a discussion in the risk analysis society whether it is fruitful or not to use risk acceptance criteria according to the principles above. It is argued that very often risk acceptance criteria are set arbitrary, and these do not necessarily support the overall best solutions. Therefore, it could be more fruitful to use some kind of risk evaluation criteria, rather than strict acceptance criteria.

In project risk management we could establish acceptance criteria related to two types of events:

- Events with severe consequences related to health, environment and safety.
- Events with severe consequences related to project costs, project quality, project duration, or even termination of the project.

In this course we will have main focus on the project costs and the duration of the project. Note that both project cost and project duration are stochastic variables and not events. Thus it is not possible to establish acceptance criteria to project cost or duration directly. Basically, there are three types of numeric values we could introduce in relation to such stochastic variables describing the project:

1. *Target.* The target expresses our ambitions in the project. The target shall be something we are striving at, and it should be possible to reach the target. It is possible to introduce (internal) bonuses, or other rewards in order to reach the targets in a project.
2. *Expectation.* The expectations are the value the stochastic variables will achieve in the long run, or our expectation about the outcome. The expectation is less ambitious than the target. The expectation will in a realistic way account for hazards, and threats and conditions which often contribute to the fact that the targets are not met.
3. *Commitment.* The commitments are values related to the stochastic variables which are regulated in agreements and contracts. For example it could be stated in the contract that a new bridge shall be completed within a given date. If we are not able to fulfil the commitments, this will usually result in economical consequences, for example penalties for defaults, or in the worst case canceling of the contract.

Problem 2.1
Discuss targets, expectations and commitments related to a new railway track between two big cities in Norway. □

We sometimes also want to discuss the *uncertainty* in e.g. the project costs. In Section 2.3.3 we have discussed the uncertainty concept in relation to project duration and costs.

2.2 Risk identification

In order to establish a risk picture three important questions are put forward:

1. What could go wrong?
2. How likely is it?
3. And if it goes wrong, how serious is it (the consequences)?

With respect to risk identification it is the first question we will answer. Since risk adresses all thre questions, it would have been better to use terms like 'hazards-' and 'threats identification' rather than the term 'risk identification', but the common practice is to use the term 'risk identification when it comes to identificaiton of what could go wrong. The risk identification could be adressed from different angles:

- A listing of undesired events.
- A listing of scenarios.
- A listing of hazards.
- A listing of threats.

An *undesired* event is an event which might occur, e.g., a large water leakage in a tunnel. A *scenario* is a description of a imagined sequence or chain of events, e.g., we have a water leakage, and we are not able to stop this leakage with ordinary tightening medium due to the possible environmental aspects which is not clarified at the moment. Further the green movement is also likely to enter the scene in this case. A *hazard* is typically related to energies, poisonous media etc, and if they are released this will result in an accident or a severe event. A *threat* is a wider term than hazard, and we include also aspects as "wrong" method applied, "lack of competence and experience". The term threat is also very often used in connection with security problems, e.g., sabotage, terrorism, and vandalism.

Problem 2.2
List examples of "undesired events", "scenarios", "hazards" and "threats" in relation to building a new railway track between two major cities in Norway. □

There exist several methods that could be used in order to identify undesired events and threats, e.g.:

- Preliminary Hazard Analysis (PHA). PHA is used to establish threats in an early phase of a project. The method will usually require some project breakdown, e.g Work Breakdown Structure (WBS) or Cost Breakdown Structure (CBS), project phases or similar. A detailed project description is usually not available at this moment.

- Task analysis (TA) and Hazard and Operability Study (HAZOP) are used on a more detailed level where we have knowledge about the various tasks.

- Use of experience data means that we try to identify events and threats based on systematic analysis of experience from the past, i.e. what have gone wrong in earlier projects.

- Checklists. Checklists exist on different levels, and could either be used in a separate analysis, or as an aid in another method, e.g. in a PHA. The checklists should, however, be put a way initially since introducing checklists early will often prevent the process of revealing project specific conditions. A checklist is primarily a list to be used in the end of the process to ensure that the "obvious" elements have not been overlooked. In Table 2.6 a such generic list of risk factors is provided. For each risk factor in the list, also some "cues" are listed which could be used to assess the significance of the risk factor in a given project. Note that this list is on a very general level, and not specific to e.g. a construction project, a tunnel project and so on.

2.3 Structuring and modelling of risk

In Section 2.2 we have identified methods to identify events and threats. We now want to relate these events and threats to the explicit models we have for project costs and project duration.

2.3.1 Model for project execution time/schedule modelling

When analysing the execution time for a project we will have a project plan and typically a Gantt diagram as a starting point. The Gantt diagram is transformed into a so-called flow network. where the connections between the activities are explicitly described. Such a flow network also comprises description of duration of the activities in terms of probability statements. The duration of each activities are stochastic variables, which we denote T_i for activity i. In a flow network we might also have uncertain activities which will be carried out only under special conditions. These conditions could be describe in terms of events, and we need to describe the probability of occurrence of such events. Thus, there is a set of quantities, i.e. time variables and events in the model. The objective is now to link the undesired events and threats discussed in Section 2.2 to these time variables and events. Time variables are described by a probability distribution function. Such a distribution function comprises parameters that characterise the time variable. Often a parametric probability distribution is described by the three quantities L (low), M (most likely) and H high. If an undesired event occur, it is likely that the values of L, M and H will be higher than in case this event does not occur. A way to include the result from the risk identification process is then to express the different values of L, M and H depending on whether the critical event occurs or not. If we in addition are able to assess the probability of occurrence of the critical event, the knowledge about this critical event has been completely included into the risk model. Based on such an explicit modelling of the critical event, we could also easily update the model in case of new information about the critical event is obtained, for example new information could be available at a later stage in the process and changes of the plan could still be possible in light of the new information.

2.3.2 Cost modelling

The cost model is usually based on the cost breakdown structure, and the cost elements will again be functions of labor cost, overtime cost, purchase price, hour cost of renting equipment, material cost, amount of material etc. The probabilistic modelling of cost is usually easier than for modelling project execution time. The principle is just to add a lot of cost terms, where each cost term is the product of the unit price and the number of units. We introduce price and volume as stochastic variables to describe the unit price and the number of units. The price and volume variables should also be linked to the undesired events and threats we have identified in Section 2.2. Often it is necessary to link the cost model to the schedule model. For example in case of delays it might be necessary to put more effort into the project to catch up with the problems, and these effort could be very costly. Also, if the project is delayed we may need to pay extra cost to sub-contractors that have to postpone their support into the project.

2.3.3 Uncertainty in schedule and cost modelling

As indicated above we will establish probabilistic models to describe the duration and cost of a project. The result of such a probabilistic modelling is that we treat the duration and cost as stochastic variables. Since duration and costs are stochastic variables, this means that there is uncertainty regarding the values they will take in the real project we are evaluating. Sometimes we split this uncertainty into three different categories, *i*) aleatory uncertainty (variability due to e.g. weather conditions, labour conflicts, breakdown of machines etc.), *ii*) parameter or epistemic uncertainty due to lack of knowledge about "true" parameter values, and *iii*) model uncertainty due to lack of detailed, or wrong modelling. Under such a thinking, the aleatory uncertainty could not be reduced, it is believed to be the result of the variability in the world which we cannot control. Uncertainty in the parameters is, however, believed to be reducible by collecting more information. Also uncertainty in the models is believed to be reducible by more detailed modelling, and decomposition of the various elements that go into the model. It is appealing to have a mental model where the uncertainty could be split into one part which we might not reduce (variability), and one part which we might reduce by thorough analysis and more investigation (increased knowledge). If we are able to demonstrate that the part of the uncertainty related to lack of knowledge and understanding has been reduced to a sufficient degree, we could then claim high confidence in the analysis. In some situation the owner, or the authorities put forward requirements which could be interpreted as confidence regarding the quality of the analysis. It is though not always clear what is meant by such a confidence level. As an example, let $E(C)$ be the expected cost of a project. A confidence statement could now be formulated as "The probability that the actual project cost is within an interval $E(C) \pm 10\%$ should at least be 70%". It is, however, not straight forward to document such a confidence level in a real analysis. The "Succesive process (trinnvisprosessen)" [4] is an attempt to demonstrate how to reduce the "uncertainty" in the result to a certain level of confidence.

We also mention that Aven [12] has recently questioned such an approach where there exist model uncertainty and parameter uncertainty, and emphasises that we in the analysis should focus on the observable quantities which will become evident for us if the project is executed, e.g. the costs, and that uncertainty in these quantities represent the lack of knowledge about which values they will take in the future. This discussion is not pursuit any more in this presentation.

Problem 2.3 Discuss different type of uncertainties in a tunnel project. Propose a classification, and identify uncertainty elements that could be reduced by *i*) further physical investigation, and *ii*) by further analysis. Also list uncertainty elements that could not be reduced before the project is actually executed. □

2.4 Risk elements for follow up: Risk and opportunity register

As risk elements and threats are identified in Section 2.2 these have to be controlled as far as possible. It is not sufficient to identify these conditions and model them in the schedule and cost models, we also have to mitigate the risk elements and threats. In order to ensure a systematic follow up of risk elements and threats it is recommended to establish a so-called *threat log*. The terms 'Risk Register' and 'Risk & Opportunity Register' (R&OR) is sometimes used rather than the term 'threat log'.

A R&OR is best managed by a database solution, for example an MS-Access database. Each row in the database represents one risk element or threat. The fields in such a database could vary, but the following fields seems reasonable:

- *ID.* An identifier is required in order to keep track of the threat in relation to the quantitative risk models, to follow up actions et.
- *Description.* A description of the threat is necessary in order to understand the content of the the problem. It could be necessary to state the immediate consequences (e.g. occupational accident), but also consequences in terms of the main objectives of the project, e.g., time and costs.
- *Likelihood or probability.* A judgment regarding how probable it is that the threat or the risk condition will be released in terms of e.g. undesired or critical events.
- *Impact.* If possilbe, give a direct impact on cost and schedule if the event occurs, either by an expected impact, or by L, M and H values.

- *References to cost and schedule.* In order to update the schedule and cost models it is convenient to give an explicit reference from the R&OR into the schedule and cost models.
- *Manageability.* Here it is descried how the threat could be influenced, either by implementing measures to eliminate the threat prior to it reveals it self, or measures in order to reduce the consequences in case of the threat will materialize.
- *Alert information.* It is important to be aware of information that could indicate the development of the threat before it eventually will materialize. If such information is available we could implement relevant measures if necessary. For example it could be possible to take ground samples at a certain cost, but utilising the information from such samples could enable us to choose appropriate methods for tunnel penetration.
- *Measures.* List of measures that could be implemented to reduce the risk.
- *Deadline and responsible.* Identification of who is responsible for implementing and follow up of the measure or threat, and any deadlines.
- *Status.* Both with respect to the threat and any measure it is valuable to specify the development, i.e. did the treat reveal it self into undesired events with unwanted consequences, did the measure play any positive effect etc.

Problem 2.4
Consider threats and risk conditions in Problem 2.2 and discuss the possibilities to mitigate the threats, and if any prior information could be available. □

2.5 Correction and controll

As the project develops the R&OR is the primary controll tool for risk follow up. By following the status of the various threats, risk elements and measures we could monitor the risk in the project. This information should of course be linked to the time and cost plans. If a given threat does not reveal in terms of undesired events, the time and cost estimates could be lowered and this gain could be utilised in other part of the project, or in other projects. In the opposite situation it is necessary to increase the time and cost estimates, and we need to consider new measures, and maybe spend some of the reserves to catch up in case of an expected delay.

During the life cycle of a project it will occur new threats and risk elements which we did not foresee in the initial risk identification process. Such threats must continuously be entered into the R&OR, and measures need to be considered.

2.6 Collection and analysis of experience - learning

After project execution it is valuable to systemise the information and knowledge we have achieved during the project. The most important data sources will be

- The risk and opportunity register (R&OR).
- Schedule plans, and the relation between plans and reality.
- Cost plans, and account numbers

Problem 2.5
Identify other sources of information that could be relevant. □

It is especially two types of analyses we will conduct:

- Systemising of what went wrong, and which measures that proved to be efficient.
- Estimation of parameters which we could include in later probabilistic schedule and cost models.

The main elements of the project risk management process is shown in Figure 2.1.

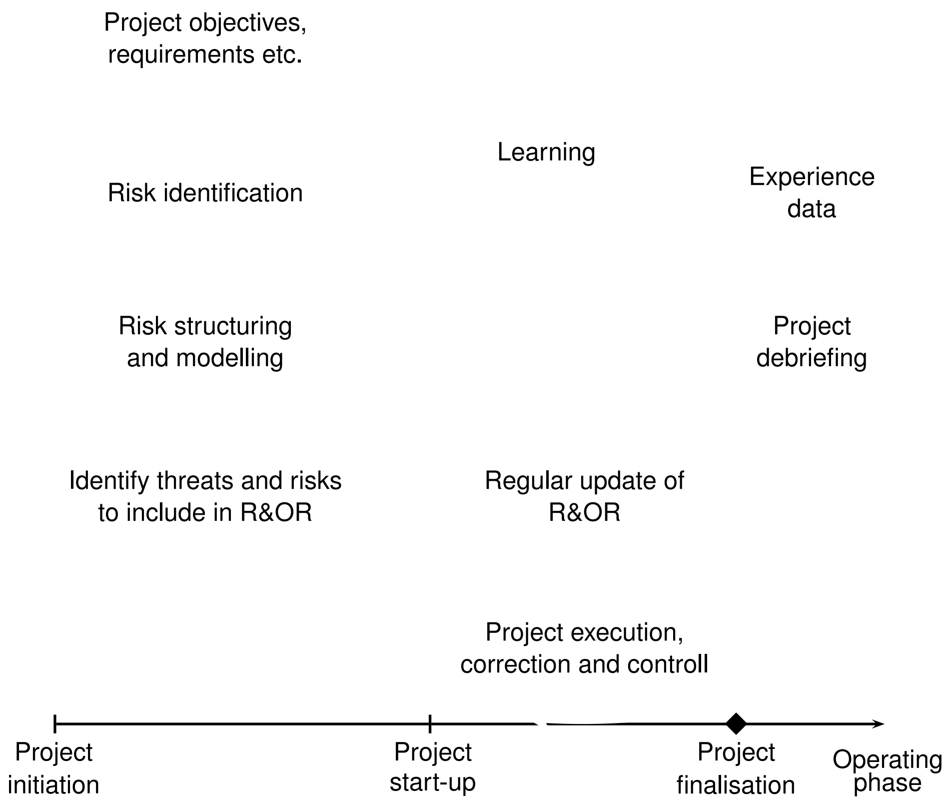

Figure 2.1: Project risk management

Table 2.1: Risk factors, adapted from State of Texas: Department of Information Resources, http://www.dir.state.tx.us/eod/qa/risk/index.htm

Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Project Fit to Customer Organization	directly supports customer organization mission and/or goals	indirectly impacts one or more goals of customer	does not support or relate to customer organization mission or goals
Project Fit to Provider Organization	directly supports provider organization mission and/or goals	indirectly impacts one or more goals of provider	does not support or relate to provider organization mission or goals
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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Customer Perception	customer expects this organization to provide this product	organization is working on project in area not expected by customer	project is mismatch with prior products or services of this organization
Work Flow	little or no change to work flow	will change some aspect or have small affect on work flow	significantly changes the work flow or method of organization
Goals Conflict	goals of projects within the program are supportive of or complimentary to each other	goals of projects do not conflict, but provide little direct support	goals of projects are in conflict, either directly or indirectly
Resource Conflict	projects within the program share resources without any conflict	projects within the program schedule resources carefully to avoid conflict	projects within the program often need the same resources at the same time (or compete for the same budget)
Customer Conflict	multiple customers of the program have common needs	multiple customers of the program have different needs, but do not conflict	multiple customers of the program are trying to drive it in very different directions
Leadership	program has active program manager who coordinates projects	program has person or team responsible for program, but unable to spend enough time to lead effectively	program has no leader, or program manager concept is not in use
Program Manager Experience	program manager has deep experience in the domain	program manager has some experience in domain, is able to leverage subject matter experts	program manager is new to the domain
Definition of the Program	program is well-defined, with a scope that is manageable by this organization	program is well-defined, but unlikely to be handled by this organization	program is not well-defined or carries conflicting objectives in the scope

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Political Influences	no particular politically-driven choices being made	project has several politically motivated decisions, such as using a vendor selected for political reasons, rather than qualifications	project has a variety of political influences or most decisions are made behind closed doors
Convenient Date	date for delivery has been set by reasonable project commitment process	date is being partially driven by need to meet marketing demo, trade show, or other mandate not related to technical estimate	date is being totally driven by need to meet marketing demo, trade show, or other mandate; little consideration of project team estimates
Use of Attractive Technology	technology selected has been in use for some time	project is being done in a sub-optimal way, to leverage the purchase or development of new technology	project is being done as a way to show a new technology or as an excuse to bring a new technology into the organization
Short Term Solution	project meets short term need without serious compromise to long term outlook	project is focused on short-term solution to a problem, with little understanding of what is needed in the long term	project team has been explicitly directed to ignore the long term outlook and focus on completing the short term deliverable
Organization Stability	little or no change in management or structure expected	some management change or reorganization expected	management or organization structure is continually or rapidly changing
Organization Roles and Responsibilities	individuals throughout the organization understand their own roles and responsibilities and those of others	individuals understand their own roles and responsibilities, but are unsure who is responsible for work outside their immediate group	many in the organization are unsure or unaware of who is responsible for many of the activities of the organization

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Policies and Standards	development policies and standards are defined and carefully followed	development policies and standards are in place, but are weak or not carefully followed	no policies or standards, or they are ill-defined and unused
Management Support	strongly committed to success of project	some commitment, not total	little or no support
Executive Involvement	visible and strong support	occasional support, provides help on issues when asked	no visible support; no help on unresolved issues
Project Objectives	verifiable project objectives, reasonable requirements	some project objectives, measures may be questionable	no established project objectives or objectives are not measurable
User Involvement	users highly involved with project team, provide significant input	users play minor roles, moderate impact on system	minimal or no user involvement; little user input
User Experience	users highly experienced in similar projects; have specific ideas of how needs can be met	users have experience with similar projects and have needs in mind	users have no previous experience with similar projects; unsure of how needs can be met
User Acceptance	users accept concepts and details of system; process is in place for user approvals	users accept most of concepts and details of system; process in place for user approvals	users do not accept any concepts or design details of system
User Training Needs	user training needs considered; training in progress or plan in place	user training needs considered; no training yet or training plan is in development	requirements not identified or not addressed
User Justification	user justification complete, accurate, sound	user justification provided, complete with some questions about applicability	no satisfactory justification for system
Project Size	small, non-complex, or easily decomposed	medium, moderate complexity, decomposable	large, highly complex, or not decomposable
Reusable Components	components available and compatible with approach	components available, but need some revision	components identified, need serious modification for use

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Supplied Components	components available and directly usable	components work under most circumstances	components known to fail in certain cases, likely to be late, or incompatible with parts of approach
Budget Size	sufficient budget allocated	questionable budget allocated	doubtful budget is sufficient
Budget Constraints	funds allocated without constraints	some questions about availability of funds	allocation in doubt or subject to change without notice
Cost Controls	well established, in place	system in place, weak in areas	system lacking or nonexistent
Delivery Commitment	stable commitment dates	some uncertain commitments	unstable, fluctuating commitments
Development Schedule	team agrees that schedule is acceptable and can be met	team finds one phase of the plan to have a schedule that is too aggressive	team agrees that two or more phases of schedule are unlikely to be met
Requirements Stability	little or no change expected to approved set (baseline)	some change expected against approved set	rapidly changing or no agreed-upon baseline
Requirements Completeness and Clarity	all completely specified and clearly written	some requirements incomplete or unclear	some requirements only in the head of the customer
Testability	product requirements easy to test, plans underway	parts of product hard to test, or minimal planning being done	most of product hard to test, or no test plans being made
Design Difficulty	well defined interfaces; design well understood	unclear how to design, or aspects of design yet to be decided	interfaces not well defined or controlled; subject to change
Implementation Difficulty	content is reasonable for this team to implement	content has elements somewhat difficult for this team to implement	content has components this team will find very difficult to implement
System Dependencies	clearly defined dependencies of the project and other parts of system	some elements of the system are well understood and planned; others are not yet comprehended	no clear plan or schedule for how the whole system will come together

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Response or other Performance Factors	readily fits boundaries needed; analysis has been done	operates occasionally at boundaries	operates continuously at boundary levels
Customer Service Impact	requires little change to customer service	requires minor changes to customer service	requires major changes to customer service approach or offerings
Data Migration Required	little or no data to migrate	much data to migrate, but good descriptions available of structure and use	much data to migrate; several types of data or no good descriptions of what is where
Pilot Approach	pilot site (or team) available and interested in participating	pilot needs to be done with several sites (who are willing) or with one who needs much help	only available pilot sites are uncooperative or in crisis mode already
Alternatives Analysis	analysis of alternatives complete, all considered, assumptions verifiable	analysis of alternatives complete, some assumptions questionable or alternatives not fully considered	analysis not completed, not all alternatives considered, or assumptions faulty
Commitment Process	changes to commitments in scope, content, schedule are reviewed and approved by all involved	changes to commitments are communicated to all involved	changes to commitments are made without review or involvement of the team
Quality Assurance Approach	QA system established, followed, effective	procedures established, but not well followed or effective	no QA process or established procedures
Development Documentation	correct and available	some deficiencies, but available	nonexistent
Use of Defined Development Process	development process in place, established, effective, followed by team	process established, but not followed or is ineffective	no formal process used
Early Identification of Defects	peer reviews are incorporated throughout	peer reviews are used sporadically	team expects to find all defects with testing
Defect Tracking	defect tracking defined, consistent, effective	defect tracking process defined, but inconsistently used	no process in place to track defects

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Change Control for Work Products	formal change control process in place, followed, effective	change control process in place, not followed or is ineffective	no change control process used
Physical Facilities	little or no modification needed	some modifications needed; some existent	major modifications needed, or facilities nonexistent
Tools Availability	in place, documented, validated	available, validated, some development needed (or minimal documentation)	unvalidated, proprietary or major development needed; no documentation
Vendor Support	complete support at reasonable price and in needed time frame	adequate support at contracted price, reasonable response time	little or no support, high cost, and/or poor response time
Contract Fit	contract with customer has good terms, communication with team is good	contract has some open issues which could interrupt team work efforts	contract has burdensome document requirements or causes extra work to comply
Disaster Recovery	all areas following security guidelines; data backed up; disaster recovery system in place; procedures followed	some security measures in place; backups done; disaster recovery considered, but procedures lacking or not followed	no security measures in place; backup lacking; disaster recovery not considered
PM Approach	product and process planning and monitoring in place	planning and monitoring need enhancement	weak or nonexistent planning and monitoring
PM Experience	PM very experienced with similar projects	PM has moderate experience or has experience with different types of projects	PM has no experience with this type of project or is new to project management
PM Authority	has line management or official authority that enables project leadership effectiveness	is able to influence those elsewhere in the organization, based on personal relationships	has little authority from location in the organization structure and little personal power to influence decision-making and resources

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Support of the PM	complete support by team and of management	support by most of team, with some reservations	no visible support; manager in name only
Team Member Availability	in place, little turnover expected; few interrupts for fire fighting	available, some turnover expected; some fire fighting	high turnover, not available; team spends most of time fighting fires
Mix of Team Skills	good mix of disciplines	some disciplines inadequately represented	some disciplines not represented at all
Team Communication	clearly communicates goals and status between the team and rest of organization	team communicates some of the information some of the time	rarely communicates clearly within team or to others who need to be informed
Application Experience	extensive experience in team with projects like this	some experience with similar projects	little or no experience with similar projects
Expertise with Application Area (Domain)	good background with application domain within development team	some experience with domain in team or able to call on experts as needed	no expertise in domain in team, no availability of experts
Experience with Project Tools	high experience	average experience	low experience
Experience with Project Process	high experience	average experience	low experience
Training of Team	training plan in place, training ongoing	training for some areas not available or training planned for future	no training plan or training not readily available
Team Spirit and Attitude	strongly committed to success of project; cooperative	willing to do what it takes to get the job done	little or no commitment to the project; not a cohesive team
Team Productivity	all milestones met, deliverables on time, productivity high	milestones met, some delays in deliverables, productivity acceptable	productivity low, milestones not met, delays in deliverables
Technology Match to Project	technology planned for project is good match to customers and problem	some of the planned technology is not well-suited to the problem or customer	selected technology is a poor match to the problem or customer

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Risk Factors	Low Risk Cues	Medium Risk Cues	High Risk Cues
Technology Experience of Project Team	good level of experience with technology	some experience with the technology	no experience with the technology
Availability of Technology Expertise	technology experts readily available	experts available elsewhere in organization	will need to acquire help from outside the organization
Maturity of Technology	technology has been in use in the industry for quite some time	technology is well understood in the industry	technology is leading edge, if not "bleeding edge" in nature
Design Complexity	easily maintained	certain aspects difficult to maintain	extremely difficult to maintain
Support Personnel	in place, experienced, sufficient in number	missing some areas of expertise	significant discipline or expertise missing
Vendor Support	complete support at reasonable price and in needed time frame	adequate support at contracted price, reasonable response time	little or no support, high cost, and/or poor response time

Chapter 3

Probability theory

3.1 Basic probability notation

In this chapter basic elements of probability theory are reviewed. Readers familiar with probability theory can skip this chappter. Readers which are very unfamiliar with this topic are advised to read an introductionary textbook in probability theory.

3.1.1 Event

In order to define probability, we need to work with events. Let as an example A be the event that there is an operator error in a control room. This is written:

$$A = \{\text{operator error}\}$$

An event may occur, or not. We do not know the outcome in advance prior to the experiment or a situation in the "real life". We also use the word event to denote a set of distinct events. For example the event that we get an even number when tossing a dice.

3.1.2 Probability

When events are defined, the probability that the event occurs is of interest. Probability is denoted by $\Pr(\cdot)$, i.e.

$$\Pr(A) = \text{Probability that } A \text{ occur}$$

The numeric value of $\Pr(A)$ may be found by:

- Studying the *sample space*.
- Analysing collected data.
- Look up the value in data hand books.

- “Expert judgement” [11].

The *sample space* defines all possible events. As an example let A = {It is Sunday}, B = {It is Monday}, .. , G = {It is Saturday}. The sample space is then given by $S = \{A, B, C, D, E, F, G\}$.

So-called Venn diagrams are useful when we want to analyse a subset of the sample space S. A rectangle represents the entire sample space, and closed curves such as a circle are used to represent subsets of the sample space as illustrated in Figure 3.1. In

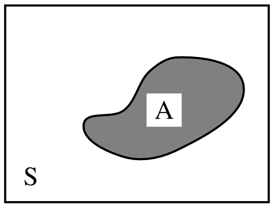
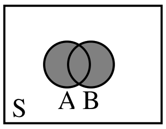
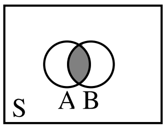


Figure 3.1: Venn diagram

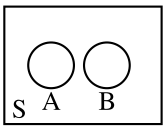
the following we will illustrate frequently used combinations of events:



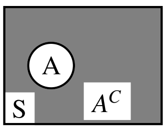
Union. We write $A \cup B$ to denote the union of A and B, i.e., the occurrence of A or B or (A and B). Let A be the event that tossing a die results in a “six”, and B be the event that we get an odd number of eyes. We then have $A \cup B = \{1, 3, 5, 6\}$.



Intersection. We write $A \cap B$ to denote the intersection of A and B, i.e. the occurrence of both A and B. As an example, let A be the event that a project is not completed in due time, and let B be the event that the budget limits are exceeded. $A \cap B$ then represent the situation that the project is not completed in due time and the budget limits are exceeded.



Disjoint events. A and B are said to be *disjoint* if they can *not* occur simultaneously, i.e. $A \cap B = \emptyset$ = the empty set. Let A be the event that tossing a die results in a “six”, and B be the event that we get an odd number of eyes. A and B are disjoint since they cannot occur simultaneously, and we have $A \cap B = \emptyset$.



Complementary events. The *complement* of an event A is all events in the sample space S except for A. The complement of an event is denoted by A^C. Let A be the event that tossing a die results in an odd number of eyes. A^C is then the event that we get an even number of eyes.

3.1.3 Probability and Kolmogorov’s axioms

Probability is a set function Pr() which maps events A_1, $A_{2,...}$ in the sample space S to real numbers. The function Pr(·) can only take values in the interval from 0 to 1, i.e. probabilities are greater or equal than 0, and less or equal than 1. Kolmogorov

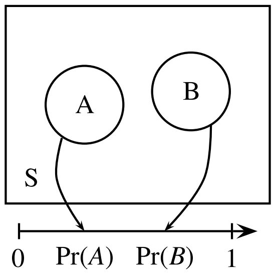


Figure 3.2: Mapping of events on the interval [0,1]

established the following axioms which all probability rules could be derived from:

1. $0 \leq \Pr(A)$

2. $\Pr(S) = 1$

3. If A_1, A_2, A_3,... is a sequence of disjoint events we shall then have:
   $\Pr(A_1 \cup A_2 \cup \ldots) = \Pr(A_1) + \Pr(A_2) + \ldots$

The axioms are the basis for establishing calculation rules when dealing with probabilities, but they do not help us in establishing numerical values for the basic probabilities $\Pr(A_1)$, $\Pr(A_2)$, etc. Historically two lines of thoughts have been established, the classical (frequentiest) and the Bayesian approach. In the classical thinking we introduce the concept of a random experiment, where $\Pr(A_i)$ is the relative frequency with which the event A_i occurs. The probability could then be interpreted as a property of the experiment, or a property of the world. By letting nature reveal itself by doing experiments, we could in principle establish all probabilities that are of interest. Within the Bayesian framework probabilities are interpreted as subjective believe about whether A_i will occur or not. Probabilities is then not a property of the world, but rather a measure of the knowledge and understanding we have about a phenomenon.

Before we set up the basic rules for probability theory that we will need, we introduce the concepts of conditional probability and independent events.

Conditional probability. $\Pr(A|B)$ denotes the conditional probability that A will occur given that B has occurred.

Independent events. A and B are said to be *independent* if information about whether B has occurred does *not* influence the probability that A will occur, i.e., $\Pr(A|B) = \Pr(A)$.

Basic rules for probability. The following calculation rules for probability apply:

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) \quad (3.1)$$

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B) \text{ if A and B are independent} \quad (3.2)$$

$$\Pr(A^C) = \Pr(A \text{ does } \textit{not} \text{ occur}) = 1 - \Pr(A) \quad (3.3)$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} \quad (3.4)$$

Example 3.1
Let the two events A and B be defined by $A = \{\text{It is Sunday}\}$ and $B = \{$It is between 6 and 8 pm).

First we note that A and B are independent but not disjoint. We will find $\Pr(A \cap B)$, $\Pr(A \cup B)$ and $\Pr(A|B)$

$$\Pr(A \cap B) = \Pr(A) \cdot \Pr(B) = \frac{1}{7} \cdot \frac{2}{24} = \frac{1}{84}$$

$$\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B) = \frac{1}{7} + \frac{2}{24} - \frac{1}{84} = \frac{9}{42}$$

$$\Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{1/84}{2/24} = \frac{1}{7}$$

□

3.1.4 The law of total probability

In many situations it is easier to assess the probability of an event B conditionally on some other events, say $A_1, A_2, \ldots, A_r$, than unconditionally. The law of total probability could then be used to assess the unconditional probability. Now, we say that $A_1, A_2, \ldots, A_r$ is a division of the sample space if the union of all A_i's covers the entire sample space, i.e. $A_1 \cup A_2 \cup \ldots \cup A_r = S$ and the A_i's are pair wise disjoint, i.e. $A_i \cap A_j = \emptyset$ for $i \neq j$. An example is shown in Figure 3.3.

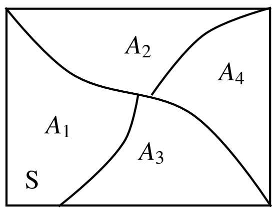


Figure 3.3: Division of the sample space

Let $A_1, A_2, \ldots, A_r$ represent a division of the sample space S, and let B be an arbitrary event in S. The law of total probability now states:

$$\Pr(B) = \sum_{i=1}^{r} \Pr(A_i) \cdot \Pr(B|A_i) \quad (3.5)$$

Example 3.2
A special component type is ordered from two suppliers A_1 and A_2. Experience has shown that components from supplier A_1 has a defect probability of 1%, whereas components from supplier A_2 has a defect probability of 2%. In average 70% of the components are provided by supplier A_1. Assume that all components are put on a common stock, and we are not able to trace the supplier for a component in the stock. A component is now fetched from the stock, and we will calculate the defect probability, Pr(B):

$$\Pr(B) = \sum_{i=1}^{r} \Pr(A_i) \cdot \Pr(B|A_i) = \Pr(A_1) \cdot \Pr(B|A_1) + \Pr(A_2) \cdot \Pr(B|A_2) =$$
$$0.7 \cdot 0.01 + 0.3 \cdot 0.02 = 1.3\%$$

□

3.1.5 Bayes rule

Now consider the example above, and assume that we have got a defect component from the stock (event B). We will derive the probability that the component originates from supplier A_1. We then use Bayes formula that states if $A_1, A_2, \ldots, A_r$ represent a division of the sample space, and B is an arbitrary event then:

$$\Pr(A_j|B) = \frac{\Pr(B|A_j) \cdot \Pr(A_j)}{\sum_{i=1}^{r} \Pr(A_i) \cdot \Pr(B|A_i)} \tag{3.6}$$

Example 3.3
We have

$$\Pr(A_1|B) = \frac{\Pr(B|A_1) \cdot \Pr(A_1)}{\sum_{i=1}^{r} \Pr(A_i) \cdot \Pr(B|A_i)} = \frac{0.01 \cdot 0.7}{0.013} = 0.54$$

Thus, the probability of A_1 is reduced from 0.7 to 0.54 when we know that the component is defect. The reason for this is that components from supplier A_1 are the best ones, and hence when we know that the component was defect, it is less likely that it was from supplier A_1. □

3.1.6 Stochastic variables

Stochastic variables are used to describe quantities which can not be predicted exactly. Note that the term *'random quantity'* is often used to denote a stochastic variable.

$$X \text{ is stochastic} \Leftrightarrow \text{ Impossible to predict the value of } X$$

To be more precise, a stochastic variable X is a real valued function that assigns a quantitative measure to each event e_i in the sample space S. Often the underlying events, e_i are of little interest. We are only interested in the stochastic variable X measured by some means. Examples of stochastic variables are given below:

- X = Life time of a component (continuous)
- R = Repair time after a failure (continuous)
- T = Duration of a construction project (continuous)
- C = Total cost of a renewal project (continuous)
- N = Number of delayed trains next month (discrete)
- W = Maintenance and operational cost next year (continuous)

Remark: We differentiate between *continuous* and *discrete* stochastic variables. Continuous stochastic variables can take any value among the real numbers, whereas discrete variables can take only a *finite* (or countable finite) number of values. □

Cumulative distribution function. A stochastic variable X is characterized by it's *cumulative distribution function*

$$F_X(x) = \Pr(X \leq x) \tag{3.7}$$

We use subscript X to emphasise the relation to the cumulative distribution function of the quantity X. The argument (lowercase x) states which values the stochastic variable X could take, or is of our interest. From the expression we observe that $F_X(x)$ states the probability that the random quantity X is less or equal than (the numeric value of) x. A typical distribution function is shown in Figure 3.4. Note that the distribution function is strictly increasing, and $0 \leq F_X(x) \leq 1$. From $F_X(x)$ we can obtain the probability that X will be within a specified interval, $[a,b]$:

$$\Pr(a < X \leq b) = F_X(b) - F_X(a) \tag{3.8}$$

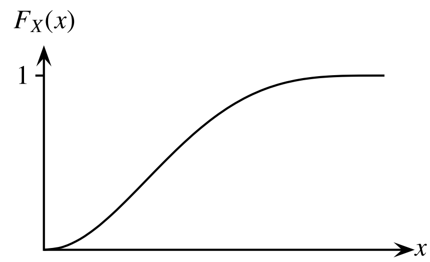


Figure 3.4: Cumulative distribution function, $F_X(x)$

Example 3.4
Assume that the probability distribution function of X is given by $F_X(x) = 1 - e^{-(0.01x)^2}$, and we will find the probability that X is in the interval (100,200]. From Equation (3.8)

we have:

$$\Pr(100 < X \leq 200) = F_X(200) - F_X(100) =$$
$$\left[1 - e^{-(0.01 \cdot 200)^2}\right] - \left[1 - e^{-(0.01 \cdot 100)^2}\right] = e^{-1} - e^{-4} \approx 0.35$$

□

Probability density function. For a continuous stochastic variable, the *probability density function* is given by

$$f_X(x) = \frac{d}{dx} F_X(x) \tag{3.9}$$

The probability density function expresses how likely the various x-values are. Note

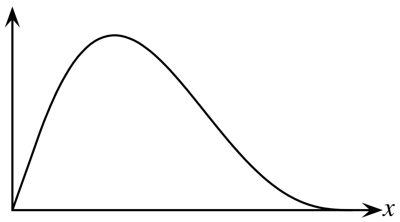


Figure 3.5: Probability density function, $f_X(x)$

that for continuous random variables the probability that X will take a specific value vanishes. However, the probability that X will fall into a small interval around a specific value is positive. For each x-value given in Figure 3.5 $f_X(x)$ could be interpreted as the probability that X will fall within a small interval around x divided by the length of this interval. Especially we have:

$$F_X(x) = \int_{-\infty}^{x} f_X(u) du \tag{3.10}$$

and

$$\Pr(a < X \leq b) = \int_{a}^{b} f_X(x) dx \tag{3.11}$$

The last expression is illustrated in Figure 3.6.

Random quantities that take discrete values are said to be discretely distributed. For such quantities we introduce the point probability for X in the point x_j:

$$p(x_j) = \Pr(X = x_j) \tag{3.12}$$

where $x_1, x_2, \ldots$ are possible values X could take.

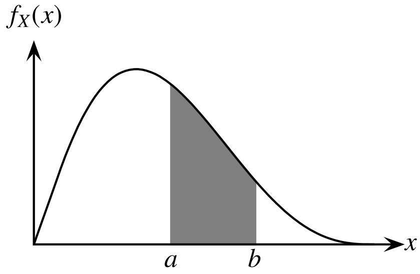


Figure 3.6: The shadded area equals $\Pr(a < X \leq b)$

Expectation. The expectation (mean) of X is given by

$$\mathrm{E}(X) = \begin{cases} \int_{-\infty}^{\infty} x \cdot f_X(x)\,dx & \text{if } X \text{ is continuous} \\ \sum_j x_j \cdot p(x_j) & \text{if } X \text{ is discrete} \end{cases} \tag{3.13}$$

The expectation can be interpreted as the long time run average of X, if an infinite amount of observations are available.

Median. The median of a distribution is the value m_0 of the stochastic variable X such that $\Pr(X \leq m_0) \geq 1/2$ and $\Pr(X \geq m_0) \geq 1/2$. In other words, the probability at or below m_0 is at least 1/2, and the probability at or above m_0 is at least 1/2.

Mode. The mode of a distribution is the value M of the stochastic variable X such that the probability density function, or point probability at M is higher or equal than for any other value of the stochastic variable. We sometimes used the term 'most likely value' rather than *mode*.

Variance. The variance of a random quantity expresses the variation in the value X will take in the long run. We denote the variance of X by:

$$\mathrm{Var}(X) = \begin{cases} \int_{-\infty}^{\infty} [x - \mathrm{E}(X)]^2 \cdot f_X(x)\,dx & \text{if } X \text{ is continuous} \\ \sum_j \left[(x_j - \mathrm{E}(X)\right]^2 \cdot p(x_j) & \text{if } X \text{ is discrete} \end{cases} \tag{3.14}$$

Standard deviation. The standard deviation of X is given by

$$\mathrm{SD}(X) = +\sqrt{\mathrm{Var}(X)} \tag{3.15}$$

The standard deviation defines an interval which observations are likely to fall into, i.e. if 100 observations are available, we expect that approximate[1] 67 of these observations

[1] This result is valid for the normal distribution. For other distributions there may be deviation from this result.

fall in the interval $[E(X) - \mathrm{SD}(X), E(X) + \mathrm{SD}(X)]$.

Precision. The precision, P, is the reciprocate of the variance, i.e. $P = \frac{1}{\mathrm{Var}(X)}$.
α-percentiles. The upper α-percentile, x_α, in a distribution $F_X(x)$ is the value satisfying $\alpha = \Pr(X > x_\alpha) = 1 - F_X(x_\alpha)$.

We end this section by giving some results regarding expectation and variances. These results apply when it is easier to express the expectation and variance of one variable if we condition on the value of another variable.

Result 3.1 Double expectation
Let X and Y be stochastic variables. We then have:

$$\mathrm{E}(X) = \mathrm{E}(\mathrm{E}(X|Y)) \tag{3.16}$$

$$\mathrm{Var}(X) = \mathrm{E}(\mathrm{Var}(X|Y)) + \mathrm{Var}(\mathrm{E}(X|Y)) \tag{3.17}$$

□

It follows easily that

$$\mathrm{E}(X) = \mathrm{E}(X|B)\Pr(B) + \mathrm{E}(X|B^C)\Pr(B^C) \tag{3.18}$$

$$\begin{aligned}\mathrm{Var}(X) = {} & \mathrm{Var}(X|B)\Pr(B) + \mathrm{Var}(X|B^C)\Pr(B^C) \\ & + [\mathrm{E}(X|B) - \mathrm{E}(X)]^2\Pr(B) + \left[\mathrm{E}(X|B^C) - \mathrm{E}(X)\right]^2\Pr(B^C)\end{aligned} \tag{3.19}$$

3.2 Common probability distributions

In this section we will present some common probability distributions. We write $X \sim$ <Name of distribution>(<parameters>) to express that X belongs to <Name of distribution>, and with parameters <parameters>. Sometimes we also use an abbreviation for the distribution, for example we write $X \sim N(3, 4)$ to express that X is normally distributed with expectation 3, and variance 4.

3.2.1 The normal distribution

X is said to be normally distributed if the probability density function of X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}}\frac{1}{\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}} \tag{3.20}$$

where μ and σ are parameters that characterise the distribution. The mean and variance are given by:

$$\begin{aligned}\mathrm{E}(X) &= \mu \\ \mathrm{Var}(X) &= \sigma^2\end{aligned} \tag{3.21}$$

The distribution function for X could not be written on closed from. Numerical methods are required to find $F_X(x)$. It is convenient to introduce a standardised normal distribution for this purpose. We say that U is standard normally distributed if it's probability density function is given by:

$$f_U(u) = \phi(u) = \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} \tag{3.22}$$

We then have

$$F_U(u) = \Phi(u) = \int_{-\infty}^{u} \phi(t)\mathrm{d}t = \int_{-\infty}^{u} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\mathrm{d}t \tag{3.23}$$

and we observe that the distribution function of U does not contain any parameters. We therefore only need one look-up table or function representing $\Phi(u)$. A look-up table is given in Table 3.1. To calculate probabilities in the non-standardised normal distribution we use the following result:

Result 3.2 If X is normally distributed with parameters μ and σ, then

$$U = \frac{X - \mu}{\sigma} \tag{3.24}$$

is standard normally distributed. □

Example 3.5 Calculation in the normal distribution
Let X be normally distributed with parameters $\mu = 5$ and $\sigma = 3$. We will find $\Pr(3 < X \leq 6)$. We have:

$$\Pr(3 < X \leq 6) = \Pr(\frac{3-\mu}{\sigma} < \frac{X-\mu}{\sigma} \leq \frac{6-\mu}{\sigma}) = \Pr(\frac{3-5}{3} < U \leq \frac{6-5}{3})$$
$$= \Phi\left(\frac{1}{3}\right) - \Phi\left(\frac{-2}{3}\right) = \Phi(0.33) - (1 - \Phi(0.67)) = 0.629 - 1 + 0.749 = 0.378$$

□

Problem 3.1 Consider the example in Example 3.5, and carry out the calculation by means of the pRisk.xls program. □

Problem 3.2 Let X be the height of men in a population, and assume X is normally distributed with parameters $\mu = 181$ and $\sigma = 4$. How large percentage of the population is more than 190 cm? □

3.2.2 The exponential distribution

X is said to be exponentially distributed if the probability density function of X is given by:

$$f_X(x) = \lambda e^{-\lambda x} \tag{3.25}$$

The cumulative distribution function is given by:

$$F_X(x) = 1 - e^{-\lambda x} \tag{3.26}$$

and the mean and variance are given by:

$$\begin{aligned} E(X) &= 1/\lambda \\ \mathrm{Var}(X) &= 1/\lambda^2 \end{aligned} \tag{3.27}$$

Note that for the exponential distribution, X will always be greater than 0. The parameter λ is often denoted the intensity in the distribution

Example 3.6
We will obtain the probability that X is greater than it's expected value. We then have:

$$\Pr(X > \mathrm{E}(X)) = 1 - \Pr(X \leq \mathrm{E}(X)) = 1 - F_X(\mathrm{E}(X)) = e^{-\lambda \mathrm{E}(X)} = e^{-1} \approx 0.37$$

□

3.2.3 The Weibull distribution

X is said to be Weibull distributed if the probability density function of X is given by:

$$f_X(x) = \alpha\lambda(\lambda x)^{\alpha-1} e^{-(\lambda x)^\alpha} \tag{3.28}$$

The cumulative distribution function is given by:

$$F_X(x) = 1 - e^{-(\lambda x)^\alpha} \tag{3.29}$$

and the mean and variance are given by:

$$\begin{aligned} \mathrm{E}(X) &= \frac{1}{\lambda}\Gamma\left[\frac{1}{\alpha} + 1\right) \\ \mathrm{Var}(X) &= \frac{1}{\lambda^2}\left(\Gamma\left(\frac{2}{\alpha} + 1\right) - \Gamma^2\left(\frac{1}{\alpha} + 1\right)\right] \end{aligned} \tag{3.30}$$

where $\Gamma(\cdot)$ is the gamma function. Note that in the Weibull distribution X will also always be positive.

3.2.4 The gamma distribution

X is said to be gamma distributed if the probability density function of X is given by:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)}(x)^{\alpha-1} e^{-\lambda x} \tag{3.31}$$

For integer values of α the gamma distribution is often denoted the *Erlang* distribution. The cumulative distribution function could then be found on closed form:

$$F_X(x) = 1 - \sum_{n=0}^{\alpha-1} \frac{(\lambda x)^\alpha}{n!} e^{-(\lambda x)} \tag{3.32}$$

For non-integer values of α numerical methods are required to obtain the cumulative distribution function. The mean and variance are given by:

$$\begin{aligned} \mathrm{E}(X) &= \frac{\alpha}{\lambda} \\ \mathrm{Var}(X) &= \frac{\alpha}{\lambda^2} \end{aligned} \tag{3.33}$$

If we know the expectation, E and the variance, V, in the gamma distribution we could obtain the parameters α and λ by: $\lambda = E/V$, and $\alpha = \lambda \cdot E$. The gamma distribution is often used as a prior distribution in a Bayesian approach.

3.2.5 The inverted gamma distribution

X is said to be inverted gamma distributed if the probability density function of X is given by:

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} \left(\frac{1}{x}\right)^{\alpha+1} e^{-\lambda/x} \tag{3.34}$$

The mean and variance are given by:

$$\begin{aligned} \mathrm{E}(X) &= \lambda/(\alpha-1) \\ \mathrm{Var}(X) &= \lambda^2(\alpha-1)^{-2}(\alpha-2)^{-1} \end{aligned} \tag{3.35}$$

Note that if X is gamma distributed with parameters α and λ, then $Y = X^{-1}$ has an inverted gamma distribution with parameters α and $1/\lambda$. If we know the expectation, E and the variance, V, of an inverted gamma distribution we could obtain α and λ by $\alpha = E^2/V + 2$, and $\lambda = E \cdot (\alpha - 1)$.

3.2.6 The lognormal distribution

X is said to be lognormal distributed if the probability density function of X is given by:

$$f_X(x) = \frac{1}{\sqrt{2\pi}} \frac{1}{\tau} \frac{1}{x} e^{-\frac{1}{2\tau^2}(\log x - \nu)^2} \tag{3.36}$$

We write $X \sim \mathrm{LN}(\nu,\tau)$. The mean and variance of X is given by

$$\begin{aligned} \mathrm{E}(X) &= e^{\nu + \frac{1}{2}\tau^2} \\ Var(X) &= e^{2\nu}(e^{2\tau^2} - e^{\tau^2}) \end{aligned} \tag{3.37}$$

The following result could be utilised:

Result 3.3 If X is lognormally distributed with parameters ν and τ, then $Y = \ln X$ is normally distributed[2] with expected value ν and variance τ^2. □

[2] $\ln(\cdot)$ is the natural logarithm function

3.2.7 The binomial distribution

Before the binomial distribution is defined, binomial trials are defined. Let A be an event, and assume that the following holds:

i) n trials are performed, and in each trial we record whether A occurrs or not.

ii) The trials are stochastic *independent* of each other.

iii) For each trial $\Pr(A) = p$

When i)-iii) is satisfied, we say that we have binomial trials. Now let X be the number of times event A occurs in such a binomial trial. X is then a stochastic variable with a binomial distribution. This is written $X \sim Bin(n, p)$.

The probability function is given by

$$P(X = x) = \binom{n}{x} p^x (1 - p)^{n-x} \text{ for } x = 1, 2, .., n \tag{3.38}$$

The cumulative distribution function $\Pr(X \leq x)$ is given in statistical tables. For the binomial distribution, expectation and variance are given by:

$$\begin{aligned} \mathrm{E}(X) &= np \\ \mathrm{Var}(X) &= np(1 - p) \end{aligned} \tag{3.39}$$

3.2.8 The Poisson distribution

The Poisson distribution is often appropriate in the situation where the stochastic variable may take the values 0,1,2,. . ., and where the expected number of occurrences is proportional to an exposure measure such as time or space. For the Poisson distribution we have the following point distribution:

$$p(x) = \Pr(X = x) = \frac{\lambda^x}{x!} e^{-\lambda} \tag{3.40}$$

For the poison distribution, expectation and variance are given by:

$$\begin{aligned} \mathrm{E}(X) &= \lambda \\ \mathrm{Var}(X) &= \lambda \end{aligned} \tag{3.41}$$

It can be proved that the Poisson distribution is appropriate if the following situation applies: Consider the occurrence of a certain event (e.g. a component failure) in an interval (a, b), and assume the following:

1. A could occur anywhere in (a,b), and the probability that A occurs in $(t, t + \Delta t)$ is approximately equal to $\lambda\Delta t$, and is independent of t (Δt should be small).

2. The probability that A occurs several times in $(t, t + \Delta t)$ is approximately 0 for small values of Δt.

3. Let I_1og I_2 be disjoint intervals in (a, b). The event A occurs within I_1 is then independent of the event A occurs in I_2.

When the criteria above are fulfilled we say we have a *Poisson point process* with intensity λ. The number of occurrences (X) of A in (a, b) is then Poisson distributed with parameter $\lambda(b-a)$, i.e.

$$p(x) = \Pr(X = x) = \frac{[\lambda(b-a)]^x}{x!} e^{-\lambda(b-a)} \tag{3.42}$$

It may also be proved that the times between occurrence of A in a Poisson point process are exponentially distributed with parameter λ.

3.2.9 The inverse-Gauss distribution

The inverse-Gauss distribution is often used when we have an "under laying" deterioration process. If this deterioration process follows a Wiener process with drift η and diffusion constant δ^2, the time T, until the first time the process reaches the value ω will be Inverse-Gauss distributed with parameters $\mu = \omega/\eta$, and $\lambda = \omega^2/\delta^2$.

If the failure progression $\Omega(t)$ follows a Wiener process it could be proven that $\Omega(t)$ - $\Omega(s)$ is normally distributed with expected value $\eta(t-s)$ and variance $\delta^2(t$ - s). That is η is the average growth rate in the curve, whereas δ^2 is an expression for the variation around the average value.

For the inverse-Gauss distribution we have:

$$F_T(t) = \Phi\left(\frac{\lambda}{\mu}\sqrt{t} - \sqrt{\lambda}\frac{1}{\sqrt{t}}\right) + \Phi\left(-\frac{\lambda}{\mu}\sqrt{t} - \sqrt{\lambda}\frac{1}{\sqrt{t}}\right)e^{2\lambda/\mu} \tag{3.43}$$

and

$$\mathrm{E}(T) = \mu \tag{3.44}$$

$$\mathrm{Var}(T) = \mu^3/\lambda \tag{3.45}$$

3.2.10 The triangular distribution

The triangular distribution has a probability density function that comprises a triangle. The lover left corner points out the lowest value (L), the upper right corner points out the highest value (H). Finally, the x-value of the third corner points out the most probable value, or mode (M).The probability density function for the triangular distribution is given by:

$$f_X(x) = \begin{cases} \frac{2(x-L)}{(M-L)(H-L)} & \text{if } L \le x \le M \\ \frac{2(H-x)}{(H-M)(H-L)} & \text{if } M \le x \le H \end{cases} \tag{3.46}$$

The cumulative distribution function is given by:

$$F_X(x) = \begin{cases} \frac{(x-L)^2}{(M-L)(H-L)} & \text{if } L \le x \le M \\ 1 - \frac{(H-x)^2}{(H-M)(H-L)} & \text{if } M \le x \le H \end{cases} \tag{3.47}$$

and the mean and variance are given by:

$$\mathrm{E}(X) = \frac{L + M + H}{3}$$
$$\mathrm{Var}(X) = \frac{L^2 + M^2 + H^2 - LM - LH - MH}{18} \tag{3.48}$$

Problem 3.3 Assume that the completion of a project is triangular distributed with parameters $L = 200$, $M = 240$ and $H = 350$. In our contract we have committed our selves to finish the project within 220 days. After 220 days we have to pay a penalty of 100 Euro per day in penalty for default. Find the total expected penalty for default in this project. □

Problem 3.4 Consider Problem 3.3 and assume that a special building method could reduce H from 350 to 300, leaving L and M unchanged. This will cost 2,000 Euro extra. Do a cost benefit analysis of this option. □

3.2.11 The PERT distribution

The PERT distribution has as the triangular distribution three parameters, L (lowest value), M (most likely value), and H (highest value). To give the probability density function for the triangular distribution we first define:

$$\alpha_1 = \frac{4M + H - 5L}{H - L}$$
$$\alpha_2 = \frac{5H - 4M - L}{H - L}$$
$$z = \frac{x - L}{H - L} \tag{3.49}$$

The probability density function is now given by:

$$f_X(x) = \frac{(x - L)^{\alpha_1 - 1}(H - x)^{\alpha_2 - 1}}{B(\alpha_1, \alpha_2)(H - L)^{\alpha_1 + \alpha_2 - 1}} \tag{3.50}$$

where $B(\cdot, \cdot)$ is the beta function. The cumulative distribution function is given by:

$$F_X(x) = \frac{B_z(\alpha_1, \alpha_2)}{B(\alpha_1, \alpha_2)} \tag{3.51}$$

where $B_z(\cdot, \cdot)$ is the incomplete beta function. The mean and variance are given by:

$$\mathrm{E}(X) = \frac{L + 4M + H}{6}$$
$$\mathrm{Var}(X) = \frac{(\mathrm{E}(X) - L)(H - \mathrm{E}(X))}{7} \tag{3.52}$$

Problem 3.5 Use the pRisk.xls program to find $\Pr(X \leq 7)$ if $X \sim \mathrm{PERT}(L = 3, M = 6, H = 10)$. □

Problem 3.6 Consider a situation where the unconditional distribution of the duration of a project groundwork activity is PERT distributed with parameters $L = 0.5$, $M = 1.5$ and $H = 3.5$ days. By a detailed analysis into the uncertainty of the situation we recognize that frozen soil is a major factor to the long duration. Let B represent the event that it is frozen soil. We now make the following assessment: Given frozen soil, the duration of the activity, $T|B$, is PERT distributed with parameters $L = 2$, $M = 2.5$ and $H = 3.5$, and if the soil is not frozen the duration of the activity, $T|B^C$, is PERT distributed with parameters $L = 0.5$, $M = 1$ and $H = 2.5$. Find $p = \Pr(B)$ such that the expectation in the conditional situation is the same as in the unconditional situation. Hint: You may use that $\mathrm{E}(T) = \mathrm{E}(T|B)\Pr(B) + \mathrm{E}(T|B^C)\Pr(B^C)$, see Equation (3.18). □

Problem 3.7 Make a sketch of the unconditional probability distribution function in the situation in Problem 3.6 when the consideration of frozen soil is taken into account. □

Problem 3.8 Find the unconditional variance of the duration in Problem 3.6. Hint: You may use Equation (3.19) □

Problem 3.9 Consider again the situation in Problem 3.6, i.e. we let in the first place $T \sim \mathrm{PERT}(L = 0.5, M = 1.5, H = 3)$. Also Let B represent frozen soil and $\Pr(B) = 0.2$. We now introduce three factors, f_B, f_{B^C} and f_V that relate the conditional situation to the original situation. The parameters relevant in the conditional situation are $\{L_B, M_B, H_B\}$ and $\{L_{B^C}, M_{B^C}, H_{B^C}\}$ in the situation where B occurs, and B does not occur respectively. We now let $M_B = f_B \cdot M$, $L_B = M_B - f_V \cdot (M - L)$, $H_B = M_B + f_V \cdot (H - M)$, $M_{B^C} = f_{B^C} \cdot M$, $L_{B^C} = M_{B^C} - f_V \cdot (M - L)$, and $H_{B^C} = M_{B^C} + f_V \cdot (H - M)$. Let $f_B = 1.5$ and $f_V = 0.5$. Find by an iterative procedure the value of f_{B^C} such that the expectation of T is equal to the original expectation. Next find f_V by a similar iterative procedure such that the variance of T is equal to the original variance. □

3.3 Assessment of parameters in parametric distributions

We have in the previous section discussed parametric probability distributions. Common for all these distributions is that they involve parameters. When using a parametric distribution, we also need to assess the parameters. In this presentation we will not discuss in detail how this could be done. If we have access to experience data, we could estimate these parameters by e.g. the maximum likelihood principle. In other situations where we have no, or very little data we would use expert judgment to assess the parameters, see e.g., [11] for further discussion on expert judgment. In this presentation we will very often assume that the uncertainty in a quantity, e.g. the duration of an activity could be described by a so-called triple estimate $\{L, M, H\}$. We will then as a general rule assume that the corresponding parametric distribution is the PERT distribution. We will further assume that the L value is the absolute minimum, and that the H value is the absolute maximum the quantity could take. It is, however, important to realise that in other presentation the L and H values are treated as lower and upper quantiles in the distribution, and often a 90% interval is assumed. This is even the situation for the PERT distribution which is defined for a *finite* domain. So if we for a given triple estimate should establish the expected value, and the standard deviation we should be carefull regarding the interpretation of the triple estimate.

3.4 Distribution of sums, products and maximum values

3.4.1 Distribution of sums

If X_1, X_2,...,X_n are random variables we might obtain the expected value, the variance and the standard deviation of the sum of the x-es:

$$\mathrm{E}(X_1 + X_2 + \ldots + X_n) = \mathrm{E}\left(\sum\nolimits_{i=1}^{n} X_i\right) = \sum\nolimits_{i=1}^{n} \mathrm{E}(X_i) \tag{3.53}$$

$$\mathrm{Var}(X_1 + X_2 + \ldots + X_n) = \mathrm{Var}\left(\sum\nolimits_{i=1}^{n} X_i\right) = \sum\nolimits_{i=1}^{n} \mathrm{Var}(X_i) \tag{3.54}$$

$$\mathrm{SD}\left(\sum\nolimits_{i=1}^{n} X_i\right) = \sqrt{\sum\nolimits_{i=1}^{n} [\mathrm{SD}(X_i)]^2} \tag{3.55}$$

Note that Equations (3.54) and (3.55) are only valid if the x-es are stochastically independent. If there is dependency between the x-es we need to include a covariance term, e.g., if we only have two variables X_1 and X_2 we have:

$$\mathrm{Var}(X_1 + X_2) = \mathrm{Var}(X_1) + \mathrm{Var}(X_2) + 2\mathrm{Cov}(X_1, X_2) \tag{3.56}$$

where $\mathrm{Cov}(X_1, X_2)$ is the covariance between X_1 and X_2.

The results above help us in determine the expectation and variance of a sum of stochastic variables, but the results could not be used to establish the probability distribution of the sum. In the following we refer some results we could utilise in many situations.

Result 3.4 Sum of normally distributed stochastic variables
Let X_1, X_2,...,X_n be independent normally distributed. Let Y be the sum of the x-es, i.e. $Y = \sum_{i=1}^{n} X_i$. Y is then normally distributed with $\mathrm{E}(Y) = \sum_{i=1}^{n} \mathrm{E}(X_i)$ and $\mathrm{Var}(Y) = \sum_{i=1}^{n} \mathrm{Var}(X_i)$. □

Result 3.5 Sum of exponentially distributed stochastic variables
Let X_1, X_2,...,X_n independent exponentially distributed with parameter λ. Let Y be the sum of the x-es, i.e. $Y = \sum_{i=1}^{n} X_i$. Y is then gamma distributed with parameters n and λ. □

Result 3.6 Sum of gamma distributed stochastic variables
Let X_1, X_2,...,X_n independent gamma distributed with parameters α and λ. Let Y be the sum of the x-es, i.e. $Y = \sum_{i=1}^{n} X_i$. Y is then gamma distributed with parameters $n\alpha$ and λ. □

Result 3.7 Central limit theorem
Let $X_1, X_2, \ldots, X_n$ be a sequence of identical independent distributed stochastic variables with expected value μ and standard deviation σ. As n approaches infinity, the average value of the x-es will asymptotically have a normal distribution with expected value μ and standard deviation $\sigma/\sqrt{n}$. Similarly, the sum of the x-es will asymptotically have a normal distribution with expected value $n\mu$ and standard deviation $\sigma\sqrt{n}$. □

Several generalizations for finite variance exist which do not require identical distribution but incorporate some conditions which guarantee that none of the variables exert a much larger influence than the others. Two such conditions are the Lindeberg condition and the Lyapunov condition. Now, as n approaches infinity, the sum of the x-es will asymptotically have a normal distribution with expected value $\sum_{i=1}^{n} \mathrm{E}(X_i)$ and variance $\sum_{i=1}^{n} \mathrm{Var}(X_i)$.

Problem 3.10 Consider a project consisting of n activities that follow each other in time. Let each activity have a PERT distribution with parameters $L = 3$, $M = 5$ and $H = 10$. Use the Monte Carlo simulation procedure in the pRisk.xls program to find the cumulative distribution function for the total duration of the project. Compare the result with using the Central Limit Theorem for various values of n. How large should n be in order to give a reasonable approximation by using the normal distribution? □

3.4.2 Distribution of a product

If X_1, $X_2, \ldots, X_n$ are *independent* stochastic variables we might obtain the expected value, the variance and the standard deviation of the product of the x-es:

$$\mathrm{E}(X_1 \cdot X_2 \cdot \ldots \cdot X_n) = E\left(\prod_{i=1}^{n} X_i\right) = \prod_{i=1}^{n} E(X_i) \tag{3.57}$$

The results for the variance and standard deviation is more complicated, and we only present the results for n=2.

$$\mathrm{Var}(X_1X_2) = \mathrm{Var}(X_1)\mathrm{Var}(X_2) + \mathrm{Var}(X_1)\,[E(X_2)]^2 + \mathrm{Var}(X_2)\,[E(X_1)]^2 \tag{3.58}$$

$$\mathrm{SD}(X_1X_2) = \sqrt{\mathrm{Var}(X_1)\mathrm{Var}(X_2) + \mathrm{Var}(X_1)[E(X_2)]^2 + \mathrm{Var}(X_2)[E(X_1)]^2} \tag{3.59}$$

Problem 3.11 Show that Equation (3.58) is correct by using the fact that $\mathrm{Var}(X) = E(X^2) - [E(X)]^2$. □

Problem 3.12 Use the program pRisk.xls to simulate the mean and standard deviation of the product X_1X_2 if both X_1 and X_2 are independent and normally distributed with expected value 10 and standard deviation 2. Compare the result with the exact result. □

3.4.3 Distribution of maximum values

Let $X_1$and X_2 be independent stochastic variables, and let $Y = \max(X_1, X_2)$. The cumulative distribution function of Y is given by:

$$\begin{aligned} F_Y(x) &= \Pr(Y \le x) = \Pr(X_1 \le x \cap X_2 \le x) \\ &= \Pr(X_1 \le x)\Pr(X_2 \le x) = F_{X_1}(x)F_{X_2}(x) \end{aligned} \tag{3.60}$$

In this situation we could easily obtain the distribution of the maximum of two stochastic variables, but it is not so easy to obtain the expectation and variance. However, since the probability density function, $f_Y(x)$ is the derivative of $F_Y(x)$ we find:

$$\mathrm{E}(Y) = \int_{-\infty}^{\infty} x \cdot f_Y(x)\,dx = \int_{-\infty}^{\infty} x \cdot \left[f_{X_1}(x)F_{X_2}(x) + f_{X_2}(x)F_{X_1}(x)\right]\,dx \tag{3.61}$$

$$\mathrm{Var}(Y) = \int_{-\infty}^{\infty} [x - E(Y)]^2 \cdot \left[f_{X_1}(x)F_{X_2}(x) + f_{X_2}(x)F_{X_1}(x)\right]\,dx \tag{3.62}$$

Problem 3.13 Find the expectation and standard deviation of $Y = \max(X_1, X_2)$ if X_1 and X_2 are independent and normally distributed with $\mu_1 = E(X_1) = 10$, $\mu_2 = E(X_2) = 7$, $\sigma_1 = \mathrm{SD}(X_1) = 2$, and $\sigma_2 = \mathrm{SD}(X_2) = 3$. Hint: You might use the routine for numerical integration implemented in the pRisk.xls program. □

Problem 3.14 Consider the problem above, but now find the result by using the Monte Carlo simulation procedure in the pRisk.xls program. □

Problem 3.15 Consider the problem above, but now find the result by using the EMax and VarMax functions in the pRisk.xls program. □

Table 3.1: The Cumulative Standard Normal Distribution

$$\Phi(z) = \Pr(Z \leq z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{u^2}{2}} du$$

z	.00	.01	.02	.03	.04	.05	.06	.07	.08	.09
0.0	.500	.504	.508	.512	.516	.520	.524	.528	.532	.536
0.1	.540	.544	.548	.552	.556	.560	.564	.567	.571	.575
0.2	.579	.583	.587	.591	.595	.599	.603	.606	.610	.614
0.3	.618	.622	.626	.629	.633	.637	.641	.644	.648	.652
0.4	.655	.659	.663	.666	.670	.674	.677	.681	.684	.688
0.5	.691	.695	.698	.702	.705	.709	.712	.716	.719	.722
0.6	.726	.729	.732	.732	.739	.742	.745	.749	.752	.755
0.7	.758	.761	.764	.767	.770	.773	.776	.779	.782	.785
0.8	.788	.791	.794	.797	.800	.802	.805	.808	.811	.813
0.9	.816	.819	.821	.824	.826	.829	.831	.834	.836	.839
1.0	.841	.844	.846	.849	.851	.853	.855	.858	.860	.862
1.1	.864	.867	.869	.871	.873	.875	.877	.879	.881	.883
1.2	.885	.887	.889	.891	.893	.894	.896	.898	.900	.901
1.3	.903	.905	.907	.908	.910	.911	.913	.915	.916	.918
1.4	.919	.921	.922	.924	.925	.926	.928	.929	.931	.932
1.5	.933	.934	.936	.937	.938	.939	.941	.942	.943	.944
1.6	.945	.946	.947	.948	.949	.951	.952	.953	.954	.954
1.7	.955	.956	.957	.958	.959	.960	.961	.962	.962	.963
1.8	.964	.965	.966	.966	.967	.968	.969	.969	.970	.971
1.9	.971	.972	.973	.973	.974	.974	.975	.976	.976	.977
2.0	.977	.978	.978	.979	.979	.980	.980	.981	.981	.982
2.1	.982	.983	.983	.983	.984	.984	.985	.985	.985	.986
2.2	.986	.986	.987	.987	.987	.988	.988	.988	.989	.989
2.3	.989	.990	.990	.990	.990	.991	.991	.991	.991	.992
2.4	.992	.992	.992	.992	.993	.993	.993	.993	.993	.994
2.5	.994	.994	.994	.994	.994	.995	.995	.995	.995	.995
2.6	.995	.995	.996	.996	.996	.996	.996	.996	.996	.996
2.7	.997	.997	.997	.997	.997	.997	.997	.997	.997	.997
2.8	.997	.998	.998	.998	.998	.998	.998	.998	.998	.998
2.9	.998	.998	.998	.998	.998	.998	.999	.999	.999	.999
3.0	.999	.999	.999	.999	.999	.999	.999	.999	.999	.999

$\Phi(-z) = 1 - \Phi(z)$

Chapter 4

Schedule

In order to analyse the duration of a project, or a project activity we use flow network models. Visually, a flow network model is similar to a bar chart, or a gantt diagram. However, we usually indicate dependencies between activities with arrows, and the y and x axes are usually not labeled. The symbols used in a flow network used in this presentation are shown in Figure 4.1. An example flow network diagram is shown in Figure 4.2.

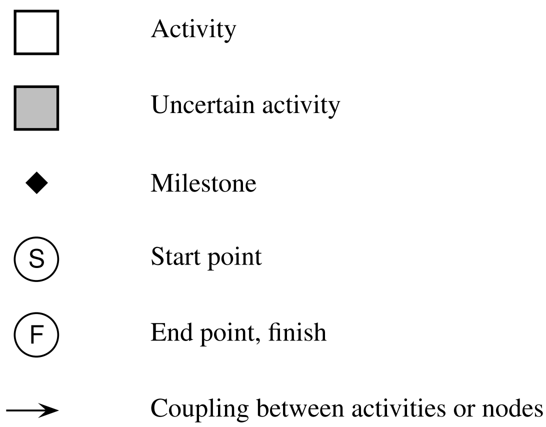


Figure 4.1: Symbols used in flow network diagrams

There exist several methods for analysing flow networks. All these models requires that the flow network is described completely in terms of dependencies between the activities. Further the duration of the activities should be described by probability distribution functions with numeric values for the parameters. When analysing such flow network we differentiate between:

- Analytical methods.

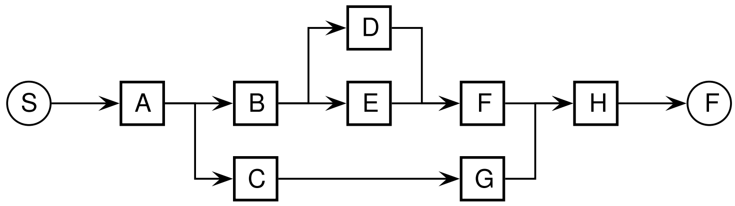


Figure 4.2: Example flow network from [3]

- Monte Carlo simulation methods.

Generally we let T denote the duration of the project we are analysing, or a part of the project, e.g. a work package. If the project comprises n activities, we often denote these activities A_i, and the duration of activity A_i is denoted T_i. Sometimes in this presentation we also use the more simplified notation where each activity is described by a letter, e.g. A, B etc. The main purpose of the schedule analysis is to establish the cumulative distribution function for the entire project duration. We might also want to establish the cumulative distribution function for parts of the project, milestones etc. Another important measure of interest is the probability that an activity will delay the project, i.e. the *criticality index*. The methods we will investigate are:

- Critical Path Method (CPM)
- Program Evaluation and Review Technique (PERT)
- Successive schedule planning (SSP)
- Monte Carlo simulation

The example diagram shown in Figure 4.2 will be used to demonstrate the various methods. This example is adapted from [3]. The parameters to describe the duration of each activity are given in Table 4.2. Fundamental for all methods is to understand the term 'path'. A path in a flow network is a set of activities from the the starting point

Activity	L (Lowest)	M (Most likely)	H (Highest)
A	2	5	9
B	4	6	9
C	7	12	21
D	5	7	10
E	4	7	11
F	2	3	6
G	3	5	9
H	5	7	10

Table 4.1: Data for the schedule demonstration example

to the end point in the network, where each activity in the set follows another activity in the set except the first activity that follows the starting point. This means that all activities in a path have to be executed in order to complete the project. Usually there are several paths in a flow network. Formally, we also include uncertain activities in a path, even if they might not be necessary to execute.

4.1 Critical Path Method (CPM)

The idea of the CPM method is to find all paths in the flow network. Next, we assume that the duration of all activities are deterministic, and typically equal to the most likely duration (M). The duration of each path is given as the sum of duration of all activities in the path. The path with the longest duration is denoted a critical path, and the duration of the project is found by the duration of the critical path (or all critical paths in case of several critical paths). In Figure 4.2 we have the following paths: $P_1 = \{A,B,D,F,H\}$, $P_2 = \{A,B,E,F,H\}$ and $P_3 = \{A,C,G,H\}$. Inserting the duration of each activity, we get the following durations $T_{P_1} = 5 + 6 + 7 + 3 + 7 = 28$, $T_{P_2} = 5 + 6 + 7 + 3 + 7 = 28$ and $T_{P_3} = 5 + 12 + 5 + 7 = 29$ for P_1, P_2 and P_3 respectively. Since P_3 has the longest duration, P_3 is a critical path, and the project duration is found to be 29. A disadvantage of the CPM method is that it cannot handle the uncertainty in the duration of each activity, i.e. it is a deterministic approach.

4.2 Program Evaluation and Review Technique (PERT)

The PERT method is similar to the CPM method when finding the project duration. We also here find the critical path, i.e. P_3. But rather than using the deterministic value for the duration, we now treat uncertainties in the activities in the critical path. The expectation and variance of a sum are given by Equations (3.53) and (3.54). If the expectations and variances are given for the various activities, we can proseed directly. However, very often the activities are described by a low (L), most likely (M), and high value (H) as in Table 4.2. If we now assume that these parameters are describing the PERT distribution, we recall that the expected value is given by $\mu = (L + 4M + H)/6$ and the variance is given by $\sigma^2 = (\mu - L)(H - \mu)/7$.

Note that the PERT method only includes one critical path. In case of more than one critical path, it will be appropriate to use the path with the highest variance. A weakness in the PERT method is that the project duration will be underestimated in case of many paths with expected duration in the same range as the critical path.

4.3 Successive schedule planning (SSP)

The idea behind the SSP-method is that we need to consider more than the critical path. To motivate for the algorithm we are going to present, consider a project with two activities A and B executed in parallel. In this situation it is reasonable to assume that the project duration will be the maximum of the duration of the two activities, and we might apply Equations (3.61) and (3.62). In this example A and B are both paths in the network, and the two paths have no common activities. In a general situation, the various paths might have common activities which complicates the calculation. Now, consider a situation with two activities B and C in parallel, where the startup of these two activities will be immediate after the finialisation of activity A. We might now establish the two paths $\{A, B\}$ and $\{A, C\}$, but we realise that the two paths have a common activity, A. For each path we could find the expectation and variance similar to what we did in the PERT method, but since the two paths share a common activity, they will not be independent, and the result for the maximum of the two paths will not be correct. In order to overcome this problem, we could in this situation first find the expectation and variance of activity A, and next add the expectation and variance for the maximum of activity B and C.

Problem 4.1 Consider the situation above with activities B and C in parallel following activity A. Further let the expectation and variance of the activity durations be given by: $\mu_A = 10, \mu_B = 7, \mu_C = 8, \sigma_A^2 = 2^2, \sigma_B^2 = 3^2$ and $\sigma_C^2 = 2^2$. Find the expectation and duration of the project by first treating the two paths $\{A, B\}$ and $\{A, C\}$ as independent. Next, carry out an exact calculation and compare the result with the first result. □

To structure the analysis we need some definitions. We define a *meeting point* where two or more arrows join before or into an activity or the endpoint. For example in Figure 4.2 the activities D and E join into a meeting point just before activity F. A *branching point* is a point where one activity is followed by two or more activities in parallel, i.e. one branch splits into two or more branches. For example in Figure 4.2 the activities B and C follow in parallel after activity A, and the branching point is just right to activity A. We also need some numerical routines for solving the integrals in Equations (3.61) and (3.62). Assume that we have access to the following routines EMax=EMax$(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$ and VarMax=VarMax$(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2)$ for solving Equations (3.61) and (3.62) respectively. Here μ_1, σ_1^2, μ_2 and σ_2^2 are expectations and variances for the two variables we are taking the maximum of. We will only consider the situation where EMax and VarMax are implemented under the assumption of independent and normally distributed variables. See pRisk.xls for such an implementation.

Problem 4.2 Show that EMax$(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) = \Delta\mu +$ EMax$(\mu_1 - \Delta\mu, \sigma_1^2, \mu_2 - \Delta\mu, \sigma_2^2)$ and VarMax$(\mu_1, \sigma_1^2, \mu_2, \sigma_2^2) =$ VarMax$(\mu_1 - \Delta\mu, \sigma_1^2, \mu_2 - \Delta\mu, \sigma_2^2)$ □

The procedure for successive schedule planning with respect to describing the project duration is now as follows[1]:

1. For each activity i, establish the expectation, μ_i and variance σ_i^2 for the duration of activity i.

2. Identify all meeting points, i.e. where one or more branches join into one arrow.

3. Repeat and follow all activities from left to right in the flow network. This process is iterative since each activity to the left of the current activity has to be processed before we can proceed.

[1]The presentation is slightly different from the original presentation by Lichtenberg (1990).

4. For each activity i establish the expected start (E_i^S), and expected finalisation (E_i^F). The expected start is equal to the expected finalisation of the preceding activity (or meeting point in case of branches are joining just before activity i). The expected finalisation is given by the expected start plus the expected duration of activity i, i.e. $E_i^F = E_i^S + \mu_i$. Note that this step cannot be executed if one or more of the activities to the left have not been processed.

5. For each activity i establish the accumulative variance, V_i^F. Here V_i^F is the accumulative variance of the activity (or meeting point) preceding activity i plus σ_i^2, i.e. $V_i^F = V_k^F + \sigma_i^2$ where k is the activity preceding activity i.

6. If there is a meeting point in the network just before the entry into an activity, we have to process this meeting point. Note that this means that two or more branches join together and the succeeding activity cannot start before all the branches, or paths up to this point, have been finalised (completed). Technically, we now introduce a virtual node at the meeting point, representing the finalisation of the two (or more) branches going into the meeting point. The virtual nodes are enumerated $V_1, V_2, \ldots$. If three or more branches join into one meeting point, we first process two branches into one virtual node, then this virtual node represent one branch which is then processed together with the third branch into another virtual node etc.

7. Let V_k be the virtual node we are processing, and assume that it is activities i and j that are joining into V_k. If one of the activities (or virtual nodes) immediate to the left of V_k has not been processed, we have to go to the left in the network until we meet processed activities or nodes. The expectation and variance for the finalisation of activity i are now given by E_i^F and V_i^F respectively. Similar we have E_j^F and V_j^F for termination of activity j. If the two paths up to activity i and j were disjoint, we could easily find the expectation and variance of the finalisation of the virtual node V_k by Equations (3.61) and (3.62), or numerically by EMax and VarMax. Typically, the two branches that join together after activity i and j did split up into two branches from one single branch prior to a branching point. Let l be the activity at which the branches did split up before joining again at the virtual node k. When finding the expectation and variance up to the virtual node k we then first find the expectation and variance up to the branching point l, and then add the expectation and variance of the maximum of the two branches from the branching point l to the virtual node k. The accumulated variance along the path from branching point l to the end of activity i is found by $V_i^{\Delta F} = V_i^F - V_l^F$. We get similar results for the other branch, i.e. the one with activity j preceding the virtual node k. The expectation and variance for the finalisation of the virtual node k is now given by $E_i^F = \text{EMax}(E_i^F, V_i^F - V_l^F, E_j^F, V_j^F - V_l^F)$ and $V_i^F = V_l^F + \text{VarMax}(E_i^F, V_i^F - V_l^F, E_j^F, V_j^F - V_l^F)$.

8. If there are more branches not processed into the meeting point, repeat until all branches are processed by creating new virtual nodes.

9. When we reach the end node, we are done.

Example 4.1
We want to demonstrate the calculation process by the flow network shown in Figure 4.2. We further assume that we have a spread sheet program available. The result of the calculations are shown in Table 4.2. In addition to the activity row, and the three rows for the low, most likely and high row, we add four rows for μ_i , σ_i^2 , E_i^F and V_i^F respectively. For each row corresponding to normal activities we calculate $\mu_i = (L + 4m + H)/6$ and $\sigma_i^2 = (\mu_i - L)(H - \mu_i)/7$. Then we calculate the expected finalisation of each activity, E_i^F as the expected finalisation of the previous activity (or virtual node) plus the expected duration of activity i, μ_i. Similarly the variance of the finalisation of activity i, V_i^F is the variance of the finilisation of the previous activity (or virtual node) plus the variance of activity i, σ_i^2. For activity A the expected finalisation and variance of the finalisation is equal to the expectation and variance of the duration of activity A since it is the first activity. We note that after activity A we have a branching point that meets again after activities F and G. For Activity B we see that $E_B^F = E_A^F + \mu_B = 5.17 + 6.17 = 11.3$, and $V_B^F = V_A^F + \sigma_B^2 = 1.73 + 0.88 = 2.61$. We note that after activity A we have a branching point that meets again after activities E and F.We proceed similarly with activities C, D and E. We now proceed to the virtual node V_1. It is convenient to insert a new row in the spread sheet program just afer activity F. In order to find the expectation and variance for this node we take advantage of the functions EMax and VarMax. The arguments to these functions are the expectation of the finalisation of each of the preceding activities, and the accumulated variance through the branches from the branching point which in this case is after activity B. $E_{V_1}^F$ = EMax(18.5,3.49-2.61,18.5,4.35-2.61)=19.14. In order to obtain the variance we use the VarMax function but we have to remember to add the accumulate variance up to finialisation of activity B, $V_{V_1}^F$ = VarMax(18.5,3.49-2.61,18.5,4.35-2.61)+2.61=3.5. We complete the sheet for the remaining activities, including the virtual node V_2. The expectation and variance of the duration of the entire project is now given by E_H^F and V_H^F respectively. □

Note that we in the SSP-method have used the PERT distribution as a basis. The method could be used for any distribution for the activities, the essential point is to assess the expectation and variance of each activity duration.

Problem 4.3 Consider Example 4.1 and carry out the calculations by your self in a spread sheet program. □

Problem 4.4 Find the cumulative distribution function for the entire project duration (T) based on the calculation in Problem 4.3, and especially find $\Pr(T > 35)$. □

4.4 Monte Carlo simulation (MCS)

The analytical models for project duration evaluation is often not flexible enough to capture relevant aspects of a project. The use of Monte Carlo simulation techniques is a supplement to analytical methods when we the situation is to complex to be analysed by analytical models. The idea of Monte Carlo simulation is that we establish

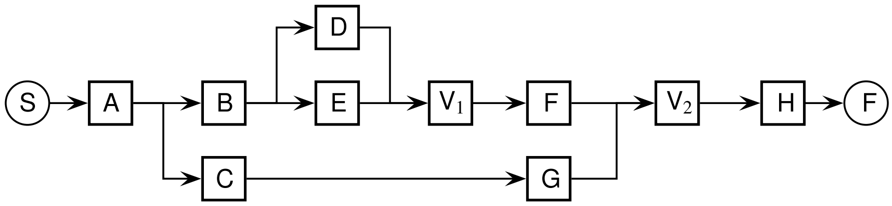


Figure 4.3: Example flow network with virutal nodes, adapted from [3]

a set of stochastic variables and events. Then we establish deterministic relations between these variables and the events, e.g. the order of which activities are executed, which activities that could be executed in parallel etc. It is important to realise that the model that describes these relations is a *deterministic model*. Such a model could be implemented in e.g. an MS Excel spreadsheet. The next idea in the Monte Carlo simulation is to generate the stochastic variables and the events (indicator variables). Most computer codes or program systems have a function that generate uniform distributed stochastic variables on the interval from 0 to 1. Given a such function it is also in principle straight forward to generate the stochastic variables we need. By inserting these stochastic variables into the deterministic model (e.g. an MS Excel model) we now get one realisation of the system, or more specific the project duration. Let t_1 be the numeric value when this process is done the first time. Now, we might repeat the process by generating another set of random quantities and insert these into the deterministic model to yield another value, say t_2. By repeating this process we could think of the generated values $t_1, t_2 \ldots$ as realisations of the project, and use the values to obtain statistical properties such as the mean, the standard deviation, the cumulative distribution function etc.

We will now illustrate how this process could be carried out with the pRisk.xls program.

Act.	L	M	H	μ_i	σ_i^2	E_i^F	V_i^F	Comment
A	2	5	9	5.17	1.73	5.17	1.73	Branching point for V_2
B	4	6	9	6.17	0.88	11.33	2.61	Branching point for V_1
C	7	12	21	12.67	7.75	17.83	8.48	
D	5	7	10	7.17	0.88	18.50	3.49	
E	4	7	11	7.17	1.73	18.50	4.35	
V_1						19.14	3.50	E & F joins, branches after B
F	2	3	6	3.33	0.51	22.48	4.01	
G	3	5	9	5.33	1.22	23.17	9.70	
V_2						24.13	5.75	G & F joins, branches after A
H	5	7	10	7.17	0.88	31.30	6.63	

Table 4.2: Data for the successive schedule planning demonstration in Example 4.1

Example 4.2
It will be convenient to establish one row in MS Excel for each activity. The first column (A) could contain the activity number, the second, third and forth (B, C and D) could then contain the parameters in the PERT distribution, similar to Table 4.2. Now we introduce three new columns (E, F and G) to contain the duration, start and finalisation of each activity respectively. We start to enter the duration of each activity. Assume that activity *A* is described in row 2 in the Excel sheet. In cell E2 we now enter the following expression for the duration:

```
=RndPert(Rand(),B2,C2,D2)
```

Here the RndPert() function is a pRisk.xls specific function, whereas the Rand() function is a standard Excel function. The procedure is repeated for all activities, and we simply copy the formula in cell E2 into the cells E3, E4 etc. We will now proceed to the start and finalisation of each activity. It will be convenient to give the cells containing the start and finalisation names in Excel. For activity *A* we give the following names D_A, S_A and F_A for the duration, start and finalisation respectively. Similarly we give the names S_B, D_B and F_B for the start, duration and finalisation of activity *B* respectively, and so on for the remaining activities. By giving name to the activities, it is easy to access them in formulas in other cells. We now use the convention cell name = expression where the cell name is the name of the cell we want to assign an expression. By inspecting the network in Figure 4.2 we easily verify the following statements for the start of the various activities:

```
S_A = 0
S_B = F_A
S_C = F_A
S_D = F_B
S_E = F_B
S_F = Max(F_D, F_E)
S_G = F_C
S_H = Max(F_F, F_G)
```

The finalisation of the activities is given as the start point plus th duration, e.g., for activity *A* we enter:

```
F_A = S_A + D_A
```

and similar for the other activities.

We have now specified the model and are prepared to simulate several runs. First we note that each time we press the F9 key Excel updates the model by generating new random numbers since we used the RAND() function in the cells containing the duration of each activity. Next we switch to the RunSimul sheet and press the Run button. □

Problem 4.5 Consider the example in Figure 4.2. We will now consider an alternative execution method for the last part of the project. Rather than executing activity H as one activity, it is possible to split this activity into two parallel activities H and I. Each of these activities could be described by the PERT distribution with parameters $L = 3$, $M = 5$ and $H = 8$. Set up the flow network for this situation, and use the pRisk.xls program to find the expectation and standard deviation of the project duration by Monte Carlo Simulation. □

4.5 Penalty for default

The contracting party might issue penalties for default to ensure that the contractor put necessary resources and effort into project execution. Penalties for default could be linked to milestones and finalisation of the entire project. In the following discussion we only consider the situation when there is defined penalty for default if the project as such is delayed. Let T be the duration of the project measured from a defined startup date. Let D be the number of days (from the startup date) before penalty for default is initiated. Finally, let PD be the size of the penalty per day. The total penalty for default is then $\max(0, (T - D)\mathrm{PD})$. The expected total penalty for default in a project is thus:

$$\mathrm{PD}_{\mathrm{Tot}} = \int_D^{\infty} (t - D)\mathrm{PD} f_T(t)\, dt \tag{4.1}$$

where $f_T(t)$ is the probability density function for the project duration.

In principal we have to perform the integration in Equation 4.1 to find the expected total penalty for default in a project. In most cases we also need to carry out numerical integration. However, if we have a Monte Carlo simulation model for the project, we might utilise that for a given project duration T the total penalty for default is $\max(0, (T - D)\mathrm{PD})$, and in e.g. pRisk.xls we could specify in the "Cell to analyse":

```
=max(0,PD*( T_End- D_Start )
```

where D_Start is the name of the cell where we have specified when penalty for default is initiated, and T_End is the name of the cell where the total project duration could be found.

Problem 4.6 Consider the example in Figure 4.2. Assume that $D = 34$, and $\mathrm{PD} = 1,000$ Euro. Find the total expected penalty for default in this project. □

4.6 Event uncertainty in the schedule model

When we describe the uncertainty about the duration of an activity by parameters such as L, M and H these parameters account for all factors and conditions that influence the duration. In some situations we might want to describe and model some important factors explicit. For example if the event W denote extremely bad weather conditions, we might describe two set of parameters $\{L, M, H\}$; one if W occurs, and one if W does not occur. That is, the duration of the activity is described by the parameters $\{L_W, M_W, H_W\}$ if W occurs, and $\{L_{W^C}, M_{W^C}, H_{W^C}\}$ if W does not occur. Next we describe the probability that W occurs by a probability statement, $p_W = \Pr(W)$.

We might now include the uncertainty about the weather (event uncertainty) explicit into the schedule model. The easiest way to include such event uncertainty is to use the Monte Carlo simulation approach. We have to do the following:

- Describe the event we want to condition on, e.g. W
- Describe the probability of the occurrence of this event, e.g. $p_W = \Pr(W)$
- Define conditional parameter statements, e.g.$\{L_W, M_W, H_W\}$ if W occurs, and $\{L_{W^C}, M_{W^C}, H_{W^C}\}$ if W does not occur for each activity duration influenced by W
- Define the event W in the pRisk.xls model
- The duration for each activity influenced by W is now entered as one expression if W occurs, and another if W does not occur.

Now, assume that it is activity A which is influenced by the event W. The following information could then be specified to pRisk.xls:

```
p_W = 0.1
Event_W = IF(Rand() < p_W,1,0)
D_A = IF(Event_W,RndPert(Rand(),5,8,12) ,RndPert(Rand(),2,5,8))
```

where the probability of bad weather conditions were set to 0.1, and we defined a cell with cell name Event_W. Finally we have used the following set of parameter values: $\{L_W, M_W, H_W\} = \{5, 8, 12\}$ and $\{L_{W^C}, M_{W^C}, H_{W^C}\} = \{2, 5, 8\}$.

Problem 4.7 Consider the example in Figure 4.2. Let W be the event {Bad weather}. Assume that this event influences primarily the activities B and C. Let $\Pr(W) = 0.1$. The parameters describing the duration of activity B and C is now similar to the situation in Problem 3.9. The transformation factors now read f_W, f_{W^C} and f_V. Let $f_W = 1.5$ and find f_{W^C} and f_V similar to the procedure in Problem 3.9. Follow this procedure both for activity B and C. Now update the Monte Carlo simulation model for the example in Figure 4.2 when the event W is introduced in the model, and compare the simulation results with the original results. □

4.7 Updating the model as we get more information

As we proceed with the project execution new information might be available, and we will know the status of activities that have been completed. At regular intervals we should therefore update the schedule model in order to optimise the effort we spend on the different activities. To update the schedule model we take the following into account

- Activities that are completed are replaced with deterministic quantities in the schedule model.
- If new activities were necessary to add to the project, these are added to the schedule model.
- If some activities were canceled these are removed from the schedule model.

- If the status of events and other risk factors that were included in the schedule model is known, we replace the probabilistic statements about these with deterministic statements.
- Other parameters (typically L, M and H) are revised in light of the knowledge available at this moment, e.g. related to resources available.

Problem 4.8 Consider the example in Figure 4.2 and the penalty for default structure as defined in Problem 4.6. assume that we now are in the project phase, and activity A has just been completed. Due to special circumstances the duration of activity A was $t_A = 10$ which is even higher than the most pessimistic assessment. First calculate the expected project duration, and total penalty for default. Next we will consider alternative production methods to increase the speed in the project. A major sub activity in activity H is to produce an element on the construction site. It is, however, possible to have this element prefabricated in advanced. The extra cost of such a prefabrication is 1,500 Euro. The gain of such a prefabrication is seen in the new distribution of the duration of activity H, i.e. we now judge $T_H \sim \text{PERT}(3, 5, 7)$. Update the pRisk.xls model and find the expected total cost with and without prefabrication. □

4.8 Examples of advanced schedule modelling

We will in the following present some situations where we need some more advanced modelling to capture the situation. This will typically require to use the Monte Carlo simulation approach, and we will use pRisk.xls to modell the situation.

Example 4.3 Moving resources between activities
We consider again the example in Figure 4.2 and will investigate the activities F and G. In expectation activity G will start slightly before activity F, whereas activity G has longer duration than H in expectation. If the situation in the project is such that activity G is not ready for start-up when activity F starts, it seems reasonable to move resources from activity F to activity G. If this is the situation, we will assume that the distributions for F and G are PERT(3,4,8) and PERT(3,4,8) respectively. To model such an operative measure in pRisk.xls we use the following statements:

```
D_F = IF(S_F>S_G,RndPert(Rand(),2,3,6),RndPert(Rand(),3,4,8))
D_G = IF(S_F>S_G,RndPert(Rand(),3,5,9),RndPert(Rand(),3,4,8))
```

When we run this model we get a slightly lower expected value for the project duration. □

Problem 4.9 Use pRisk.xls to perform the calculations in Example 4.3. □

Example 4.4 Time window
We consider again the example in Figure 4.2 and event B which represents an activity that could only be executed within certain time windows. We will assume that if activity A is not completed before time $t = 8$, we could not start up activity B before time $t = 20$. To model the effect of such a time window in pRisk.xls we use the following

statement:

```
S_B = IF(F_A>8,20,F_A)
```

□

Problem 4.10 Use pRisk.xls to perform the calculations in Example 4.4. □

Chapter 5

Decision under uncertainties

5.1 Introduction

In this section we will give a general introduction to the field of decision theory where uncertainties are involved. Examples are related to project risk management. As a motivation consider the following situation where we have to choose between two or more alternatives:

- Choice of sub contractor.
- Choice of concept for an oil production platform.
- Choice between double track and single track for a new railway line.
- Choice of tunnel trace now, or perform more investigation into the ground.

We could also have decisions related to continuous variables:

- When to make an agreement with one of the sub contractors.
- When to start preparing for a major shut-down.
- Choice of diameter for a gas pipeline towards Skogn.
- Dimension of a critical part in a new construction.

5.1.1 Overview of the method

We will first consider situations where one and only one decision is to be made. We will denote the decision with the letter d. The decision alternatives will be denoted $a_1, a_2, \ldots, a_m$. The decision, d, could then be which sub contractor to choose, and the alternative a_i is the decision that we choose sub contractor i. The result of our choice will be a set of end consequences $\mathbf{Y}$. The end consequences could occur at different times in the future, but we will simplify and assume that the effect will be immediate after the decision is made. In more complex situations we have to consider that the effect will come some time in the future, and discounting is an issue. $\mathbf{Y} = [Y_1, Y_2, \ldots, Y_r]$ is an attribute vector and comprises many dimensions. For example Y_1 could be the project duration, Y_2 could be the project costs etc. Further we note that the Y_i's are stochastic variables and the values will depend on our decision d.We will seek the decision that gives the "best" value of the attribute vector $\mathbf{Y}$. It is common to differentiate between the following four situations:

1. *Decision under certainty.* In this situation all the outcomes are known, and we will know for sure what the outcome will be for the different decision alternatives.
2. *Decision under risk.* In this situation all the possible outcomes are known, but we do not know which outcome will be the result of our decision. We are able to state probabilities for the various outcomes.
3. *Decision under uncertainty.* In this situation all the possible outcomes are known, and we are unsure about the probabilities for the various outcomes.
4. *Decision under ignorance.* In this situation we do not know all the possible outcomes, and we are also unsure about the probabilities of those outcomes we know about.

In this presentation we will only consider decisions under risk and uncertainties. We also note that many authors claim that it is not meaningful to state uncertainty about the probabilities, it is the outcome which is uncertain, not the probabilities. When it comes to the final outcome, i.e. related to the attribute vector $\mathbf{Y}$ we agree that there is no uncertainty in the probability distribution of $\mathbf{Y}$, i.e. the probability distribution contains all the uncertainty about $\mathbf{Y}$. We will therefore not differentiate between the situation of decisions under risk, and decisions under uncertainty. Most frequently we will use the term 'decision under uncertainty'.

As indicated above we will seek the decision d that gives the best "value" of the attribute vector $\mathbf{Y}$, for example the lowest cost and the shortest execution time of a project. There are, however, some difficulties in this approach:

- We will not be neutral to the risk. Very often we are willing to make a decision that do not give the maximum expected revenue, but rather choose an option with a lower expected revenue but with a lower probability of big losses. We are what is called risk averse.
- The attribute vector $\mathbf{Y}$ comprises several dimensions, and it is not straight forward how to weight these dimensions. For example how should we treat a project with low cost, but a higher risk of accidents during project execution?

In order to treat such decision situations we introduced the concept of utility theory, and utility functions. We will only briefly mention the major aspects, and refer to [9] for further discussions. First we will treat situations where we make only one decision, and the end consequences are assumed to take effect immediately after our decision. In more complex situations the effects will come on a later stage, and we could make several decision in a sequence with time delays between each decision. In such decision problems we often use *decision trees* to help assisting the decision process.

5.2 Basic concepts

We use the notation d about a decision where it is only one decision to be made, whereas we use the notation $d_1, d_2, \ldots$ in the situation where several decisions have to be made. In this situation we also need an extra index for the decision alternatives. In the situation where the decision is to choose a value of a numeric quantity, either discrete or continuous we will notationally not differentiate between the decision, and the decision variable (d).

5.2.1 Discrete end consequences vs attribute vector

Generally we use $\mathbf{Y}$ to describe the values of the end consequences resulting from a decision. In some situations we want to simplify the representation by a set of few end consequences, EC. We will let the end consequences, EC_j, be disjoint. Further we will let $p_j = \Pr(EC_j \text{ occurs})$ be the probability that we get end consequence EC_j. It is not always straight forward to determine what is the most convenient, either to work with the full attribute vector $\mathbf{Y}$, or the set of end consequences $EC_1, EC_2, \ldots$. This will depend on the level of precision in the risk analysis, the skill of the risk analyst, or the decision maker etc.

In order to see the difference, consider the occupational safety dimension during project execution. If we work with end consequences it could be natural to introduce the following end consequences:

1. EC_0 = No injury
2. EC_1 = Minor injury
3. EC_2 = Medical treatment
4. EC_3 = Serious injury
5. EC_4 = 1 fatality
6. EC_5 = 2-10 fatalities
7. EC_6 = > 10 fatalities

In order to describe the expected result we also specify the corresponding probabilities, $p_0, p_1, \ldots p_6$. These probabilities will be dependent on the decisions we make. If we want to use a full attribute vector we could use $\mathbf{Y} = [Y_1, Y_2, Y_3, Y_4]$, where Y_1 = number of minor injuries, Y_2 = number of major injuries, Y_3 = number of fatalities, and Y_4 = is the number of gross accidents, i.e. accidents with five or more fatalities. In this latter situation we specify the expected outcome in terms of the joint probability distribution function of $\mathbf{Y}$. We then often introduce parameters that depend on the decision d we make.

Utility function

The utility function expresses the preferences of the decision maker regarding various attribute vectors or end consequences. A prerequisite for establishing a utility function is that the decision maker is able to express preferences between different values of the attribute vector. For example in a one dimensional situation where we set Y = NPV (net present value) this will be rather obvious in the first place, it is reasonable that all decision makers will prefer a higher value to a lower value. Now let y_1 and y_2 denote two arbitrary values Y could take. The following relations are of interest between y_1 and y_2:

Relation	Explanation
$y_1 \sim y_2$	y_1 and y_2 is considered equal
$y_1 \succ y_2$	y_1 is preferred over y_2
$y_1 \succeq y_2$	y_1 is as least as preferable as y_2
$y_1 \prec y_2$	y_2 is preferred over y_1
$y_1 \preceq y_2$	y_2 is as least as preferable as y_1

The utility function is now a function that assigns a one-dimensional utility value to each value of the attribute vector or quantity, $u = u(y)$. For the utility function we require:

$y_1 \sim y_2 \Leftrightarrow u(y_1) = u(y_2)$
$y_1 \succ y_2 \Leftrightarrow u(y_1) > u(y_2)$
$y_1 \succeq y_2 \Leftrightarrow u(y_1) \geq u(y_2)$
$y_1 \prec y_2 \Leftrightarrow u(y_1) < u(y_2)$
$y_1 \preceq y_2 \Leftrightarrow u(y_1) \leq u(y_2)$

There exists, however, an infinite number of utility functions that satisfy the above criteria and we therefore want to fix the utility function for some values. In order to be useful, the utility function should also express how much we prefer e.g. y_1 over y_2. Further we also want the utility function to reflect the fact that there will be uncertainty regarding the future value of the attribute Y. We still consider the one dimensional situation where Y = NPV (net present value). Y will be a stochastic variable in the decision point. Now, assume that we could choose between a decision A that for sure gives the net present value $Y = y_0$ and the decision B that gives the net present value $Y = y_1$ with probability α and the net present value $Y = y_2$ with probability $1 - \alpha$. Further assume that $y_1 \prec y_0 \prec y_2$. For a given set of values of y_0, y_1 and y_2 there will exist a value of α which makes the decision maker indifferent between the two decisions A and B. This will be reflected in the utility function which must satisfy:

$$u(y_0) = \alpha u(y_1) + (1 - \alpha)u(y_2) \tag{5.1}$$

Equation (5.1) could now in principle be used to establish the utility function. In this process we might restrict our selves to let the utility function take values between 0 and 1, or 0 and 100.

Example 5.1 Private economy

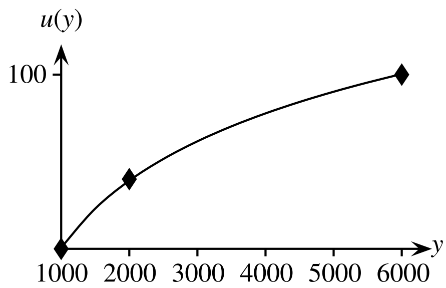


Figure 5.1: Utility function for Example 5.1

We are asked to do a job in the firm SmartConsult. It will be 100 hours of work, and we are offered two options for payment:

1. A fixed hour rate of 20 Euro per hour.
2. A baseline hour rate of 10 Euro, and an additional value of 50 Euro which will be paid if the project reaches the targets that have been set up.

We have been studying the progression in the project so far, and assess the probability that the extra 50 Euro to be paid is 40%. Simple calculations shows that the second alternative gives the highest expected hour rate (30 Euro vs 20 Euro). However, we are in a cash position which makes it very difficult for us if we only receive 10 Euro per hour. After some considerations we have found out that the two alternatives are equal, i.e. neither of them are prefered over the other.

We will now utilise Equation (5.1) to set up our utility function. Three points could be assessed, $u(1\,000)$, $u(2\,000)$, and $u(6\,000)$. Since we arbitrary may choose the end points, we let $u(1\,000) = 0$ and $u(6\,000) = 100$. We now have ($\alpha = 0.6$):

$$u(2\,000) = 0.6u(1\,000) + 0.4u(6\,000) = 40$$

The utility function is shown in Figure 5.1 where we have fitted the function $u(y) = 55.702 \ln(y) - 384.25$ and the diamonds represent the assessed values. □

Problem 5.1 Consider Example 5.1. What probability would you required for being paid the additional 50 Euro per hour in order to treat the two alternatives as equal. □

Problem 5.2 Consider Example 5.1 again, but assume that you were going to work 500 hours. Make a sketch of your utility function in this situation. □

For private economies we are usually risk averse. Risk aversion means a concave utility function as shown in Figure 5.1. Also smaller enterprises will often be risk averse reflecting that rather than optimising expected revenue, decisions are taken to minimise the probability of big losses which could lead to bankruptcy. Larger enterprises will often have an almost linear utility function (in monetary values) because their economical strength is good, and there is no real possibility for bankruptcy.

Utility function for quantities other than monetary units

In Example 5.1 we have seen how the utility function could be established for monetary units. We will now investigate how we could include the safety dimension into the utility function. Two important questions will be raised:

- What is the benefit, or utility of saving one (statistical) life vs saving 10 statistical lives?
- What is the benefit, or utility of saving one (statistical) life vs the possibility to earn an extra million Euro?

In the first situation we deal with the question to rank the consequences within the same main dimension (safety), whereas we in the second situation need to compare benefits or disadvantages across dimensions. The discussion below will be very short, and we refer to e.g. [10] for further discussion on this topic.

The first issue we will discuss is the concept 'value of prevented fatality' (VPF). The idea behind this concept is that in any industrial activity, transportation services etc there will always be a risk of accidents, and hence possibilities of severe injuries and fatalities. As a decision maker we have to face this fact. However, we will make effort to reduce this risk, and we are willing to spend money to achieve such a reduction. The VPF value states then how much we are willing to spend in order to prevent one statistical fatality. We use the term 'statistical' fatality to emphasise that this willingness to pay is not related to specific persons, but arbitrary persons where it is not meaningful or possible to identify single persons. In some presentation also the term 'value of life' (VOL) has been used. We feel that this term is not appropriate because the term indicates that the life it self has a value which could be measured in monetary units. This is not our perspective. The value of life it self could not be measured. What we could assess figures to, is what we are willing to pay in order to reduce risk, or the probability of fatalities. Hence, the term VPF make more sense in our understanding.

If we accept that the term VPF make sense, then the next question will be how to assess the value of VPF. Different approaches exist. One approach is to look into economical considerations from the society point of view. For example we could calculate the reduction in GNP (Gross National Product) caused by a fatality. Such calculations have been carried out, and in e.g. Norway this indicate a value of 3 million Euro for VPF. Another approach has been to ask single persons about their willingness to pay for risk reduction (see "The change in risk of death" [16]). For example for buyers of cars, we could ask what they are willing to pay for a given safety system or measure, for example an improved airbag system. Let assume that the amount one is willing to pay is ΔW, and that the assessed risk reduction during the service life of the car is ΔP. It would then be natural to set VPF = $\Delta W/\Delta P$. In Norway no such systematic surveys have been conducted, but more arbitrary surveys at NTNU among ordinary students and continuation students a value of 2.5 million Euro has also been found for VPF. We will emphasise some challenges of such a willingness to pay approach:

- Different persons have different preferences. For example young people tend to be less willing to pay for risk reduction compared to older persons with family obligations.

- Individuals are not consistent in their preference statements.
- In real surveys to establish ΔW and ΔP we face the problem that other dimensions than being killed are involved, e.g. the risk of minor and major injuries. Further one does not only consider the life of one self, but also the life of family members etc when making decisions about safety.
- It is not obvious that "what I am willing to pay" is what I want the society to pay for risk reduction in general, or what I expect my employer to pay for my risk reduction.

It is a tendency to set a lower value for VPF when it comes to the area of public responsibility compared to industrial activity. For example in the petroleum industry we see VPF values in the order 10 to 15 million Euro. It is also a tendency to set a higher VPF for multiple fatality accidents compared to single fatality accidents. This could be interpreted as an aversion against gross accidents. This aversion should not be confused with risk aversions which would be an aversion against a high number of fatalities in general, and not the number of fatalities in single accidents. An another perspective in this discussion is how we should treat injuries in such a framework. One common approach here is to introduce the concept of 'equivalent fatality'. For example we could be willing to pay five times more to prevent a fatality than a severe injury, which corresponds to an equivalent fatality of 0.2.

In a utility function approach we could now in case of a VPF value 2.5 million Euro let the utility of one fatality be equivalent to -2.5. If we now extend the situation to include multiple fatality accidents, and minor and major injuries, we could set up a more general utility function:

$$u(y_1, y_2, y_3, y_4) = -0.03y_1 - 0.5y_2 - 2.5y_3 - 7y_4 \tag{5.2}$$

where y_1 is the number of minor injuries, y_2 is the number of major injuries, y_3 is the number of fatalities in accidents with less than five fatalities, and y_4 is the number of fatalities in gross accidents (five or more fatalities in one accident). It is important to emphasise that the utility function offered in Equation (5.2) is a function that could be used as a start in a discussion about value trade-offs and preferences, and should not be considered as the "correct utility function". Also note that Equation (5.2) includes an aversion against gross accidents, but there is no risk aversion in terms of a concave utility function in the attributes. If we also want to include attribute y_7 as the profit in a project measured in million Euro we could extend the utility function:

$$u(y_1, y_2, y_3, y_4) = -0.03y_1 - 0.5y_2 - 2.5y_3 - 7y_4 + y_7 - ae^{-by_7} \tag{5.3}$$

where a and b are constant. Reasonable values of these constants are $a = 0.08$ and $b = 0.7$.

The utility function in Equation (5.3) is an additive utility function. Very often we use additive utility functions for simplicity. However, arguments could indicate that a situation with one extra fatality is "worse" if there is a situation with a gross accident than without such a gross accident. Such discussions will not be pursued any further, and we refer to [9].

5.2.2 Maximising expected utility

In the previous sections we have seen principles for establishing a utility function. The utility function expresses our preferences and value trade-offs. The utility function is independent of the given decision situation we are facing and could be viewed as a general function we could use in many decision situations. We also observe that the utility function is a function of the attributes. In a given situation these attributes, Y_1, Y_2, ..., are stochastic variables which also means that the utility function will be stochastic, and the idea is to choose the decision that maximises the expected utility.

Result 5.1
The optimal decision d is the decision that maximises expected utility, $\mathrm{E}(u(\mathbf{Y})) = \int_{-\infty}^{\infty} u(\mathbf{y}) f_{\mathbf{Y}}(\mathbf{y}) d\mathbf{y}$ □

The basic steps in obtaining the optimal decision is then:

1. Establish an explicit expression for the utility function, $u = u(y_1, y_2, \ldots)$ which corresponds to the preferences and value trade-offs of the decision maker.
2. Establish the probability distribution function for the attribute vector $\mathbf{Y} = [Y_1, Y_2, \ldots]$ for each decision alternative, or for each value of a decision variable (d).
3. Calculate the expected utility to each decision alternative by integrating the utility function over the probability distribution of the attribute vector.
4. Find the decision alternative that gives the maximum expected utility.

Problem 5.3 In this problem you shall first make an attempt to construct the utility function $u(y)$ for a given decision maker. In the problem there is only one dimension, and the attribute y is measured in thousand Euro by the procedures we have established in the previous sections. Assume that $u(-100) = 0$, and $u(400) = 1$.

a) Why do we have the freedom to assess two points on the utility function, and why is is suitable to use these two values.

Now, assume that the decision maker makes the following considerations regardingthe outcome of a project:

- An uncertain project which gives -100 with probability 0.50 and +400 with probability 0.50 is considered as equal attractive as receiving the fixed amount +150.
- An uncertain project which gives -100 with probability 0.50 and +150 with probability 0.50 is considered as equal attractive as receiving the fixed amount +100.
- An uncertain project which gives +150 with probability 0.50 and +400 with probability 0.50 is considered as equal attractive as receiving the fixed amount +225.

b) Draw the points on the utility function which you could calculate based on the above information, and make a sketch of the utility function in the interval -100 to +400.

c) What does the graph say about the decision makers attitudes to risk?

d) Use the graph to choose the optimum project among the following projects:

A) A project returning -100 with probability 0.2, +150 with probability 0.2 and +350 with probability 0.6.

B) A project returning 0 with probability 0.4 and +400 with probability 0.6.

e) Which of these two projects would the decision maker choose if he adopts the principle of maximum expectation. □

Problem 5.4 In a tunnel project one could choose between bursting or drilling. Bursting is considered to be the cheapest alternative, but the risk of personal injuries or fatalities is considered higher. Assume the utility function given in Equation 5.2 on page 65. Let $f_i = \mathrm{E}(Y_i), i = 1, \ldots, 4$ be the expected number of minor injuries, serious injures etc. and assume the following numbers:

- Bursting $[f_1,f_2,\ldots,f_4]$ = [10, 1, 0.03,0.008]

- Drilling $[f_1,f_2,\ldots,f_4]$ = [7, 0.2, 0.01,0.001]

How much cheaper need bursting be compared to drilling if these two methods should be equally valued with respect to utility? □

Problem 5.5 A company has established the following utility function: $u(y) = y - ae^{-by}$, where $a = 1/5$ and $b = 1/2$, to be applied for prioritization of projects. y is given in million NOKs and represents the profit in the projects.

1. Make a sketch of the utility function, and discuss the risk attitude of the company.

2. The company considers to invest in a project which (i) either gives a profit of 10 (million NOKs), or (ii) a loss of 10 (million NOKs). What probability of success is required to invest in the project?

3. The company is going to choose between two projects, A and B. By analysis the following has been derived: $Y_A \sim N(6, 42)$ and $Y_B \sim N(6.1, 52)$, where Y is the profit, and $N()$ indicates normally distributed quantities. Calculate the expected utility for each project to decide which project is best. Discuss the result. Hint: Show that if Y is normally distributed with expected value μ and standard deviation σ, then the expected utility is found by use of moment generating function to be: $\mathrm{E}(u(Y)) = \mu - ae^{-b\mu+1/2b^2\sigma^2}$ provided the utility function is given by u(y) = $u(y) = y - ae^{-}by$.

□

5.2.3 Examples with one decision node

In this section we will investigate examples where only one decision is going to be made.

Example 5.2 Maximising expected utility - private economy
In Example 5.1 we established the utility function in a situation with private economy. The function could be written as: $u(y) = 55.702\ln(y) - 384.25$. Now, assume that we are offered a job with two different forms of payment. For both alternatives the possible amounts are:

- $Y = Y_L = 2{,}000$
- $Y = Y_M = 5{,}000$
- $Y = Y_H = 8{,}000$

However, there is a difference in the probabilities for the two alternative forms of payments. These are shown under the column p for alternative a_1 and a_2 in Table 5.2 respectively. Expected utility is found by:

	Alternative a_1			Alternative a_2		
↓ Amount	p	U	V	P	U	V
2,000	0.1	3.9	200	0.3	11.7	600
5,000	0.8	72.1	4,000	0.4	36.1	2,000
8,000	0.1	11.6	800	0.3	34.9	2,400
Sum→	1.0	87.7	5,000	1.0	82.7	5,000

$$E(u(Y)) = \sum_{y\in\{Y_L,Y_M,Y_H\}} u(y)\Pr(Y = y) = \sum_{y\in\{Y_L,Y_M,Y_H\}} (55.702\ln(y) - 384.25)\Pr(Y = y) \quad (5.4)$$

where each term in the sum is calculated in the column for U in Table 5.2. In the V column we have similarly calculated the expected monetary value. In the last row the sum is shown, and we observe that the expected utility for a_1 and a_2 is 87.7 and 82.7 respectively, and hence alternative a_1 has the largest expected utility. When expectations are considered, the two alternatives are equivalent.

Problem 5.6 Consider Example 5.2, but now assume that the probability distribution for the payments are PERT(4000,5000,6000) and PERT(2000,6000,8000) respective. Hint: you might use the program pRisk.xls. □

Example 5.3 Tender offer - linear utility function
We are going to prepare a tender for a large road project. We have made a cost estimate, and found the total cost of executing the project, PC, to be PERT distributed, PC ∼ PERT(L, M, H) = PERT(10 , 30 ,80) where all costs are given in million Euro. We also have some knowledge about our competitors. We have judged the lowest tender

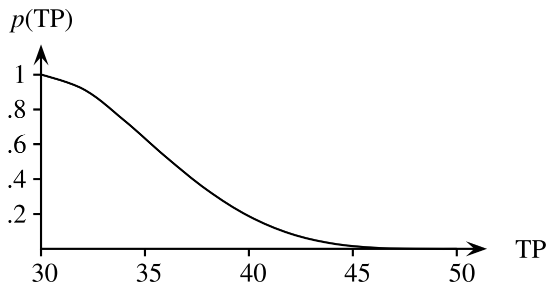


Figure 5.2: Probability p of getting the project as a function of tender price, TP

from the other contractors, LP, to be PERT distributed, e.g. LP ~ PERT(30, 35, 50). The challenge is to set our tender price (TP) such that it is lower than the lowest of the (serious) competitors , but also not too low as we then will loose money. The following notation is introduced:

TP = Our tender price, TP, i.e. the decision variable.

LP = Lowest tender prince among our competitors. We assume a lump sum contract.

PC = Project cost, i.e. a stochastic variable.

Y_1 = 1 if we get the contract, 0 otherwise, i.e. $Y_1 = I_{LP>TP}$.

p = The probability that we get the contract, $p = p(\text{TP}) = E(Y_1) = \Pr(\text{LP} > \text{TP})$.

Y_2 = TP = Our tender price. TP is also an "attribute" because this is our income.

Y_3 = PC = project cost.

u = utility function, $u(y_1, y_2, y_3) = y_1 \cdot (y_2 - y_3)$.

Expected utility is given by

$$\text{E}(u(Y_1, Y_2, Y_3|\text{TP})) = \text{E}(Y_1) \cdot \text{E}(Y_2 - Y_3) = p(TP) \cdot (\text{TP} - \text{E(PC)}) \tag{5.5}$$

In order to find p = (TP) we could utilise the pRisk.xls program, and the function CDFPert. The syntax to enter in an EXCEL cell where we store the result is:

```
= 1 - CDFPert(TP,30,35,50)
```

where TP is a cell reference or a numeric value for the tender price. Figure 5.2 shows the probability of getting the contract as a function of the tender price. Given that we get the contract, the expected utility equals TP-E(PC). The expectation in the PERT distribution is given as $(L + 4M + H)/6$, i.e. $(10 + 4 \cdot 30 + 80)/6 = 35$ million Euro, and the expected utility equals:

$$\text{E}(u(Y_1, Y_2, Y_3|\text{TP})) = p(\text{TP}) \cdot (\text{TP} - 35) \tag{5.6}$$

By using the result for $p = p(\text{TP})$ from Figure 5.2 we could easily find the expected utility as shown in Figure 5.3. The optimum tender price is found to be 39 million Euro. □

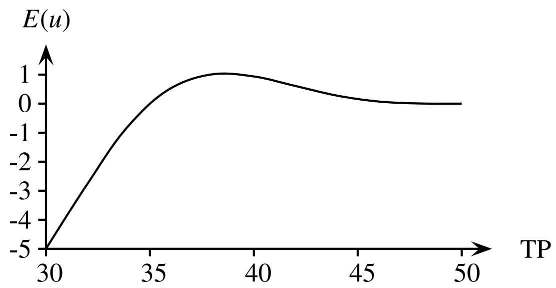


Figure 5.3: Expected utility as a function of the tender price, TP

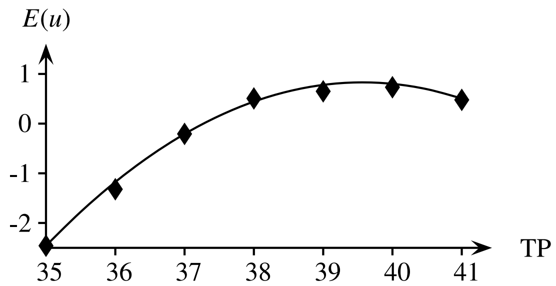


Figure 5.4: Expected utility as a function of the tender price, TP

Example 5.4 Tender offer - concave utility function
We will investigate the situation in Example 5.3 but we will now assume that the decision maker is risk averse. A possible utility function is:

$$u(y) = y - a \cdot e^{-by} \tag{5.7}$$

where a and b er parameters. We set $a = 0.2$, and $b = 0.2$, and we let money be measured in million Euro. With these parameters the utility of 10 million Euro is then 9.97 whereas the utility of a loss of 20 million Euro is - 30.9 and the utility of -30 is -110. In this example we will also add one if we get the contract. This extra utility unit could represent the value of competence improvement. The utility function is then:

$$u(y_1, y_2, y_3) = y_1 \cdot (y_2 - y_3) - 0.2 \cdot exp(0.2 \cdot y_1 \cdot (y_2 - y_3)) + y_1 \tag{5.8}$$

It will not be easy to maximise expected utility as a function of the tender price, TP. We therefore use pRisk.xls to carry out a Monte Carlo simulation, and the result is shown in Figure 5.4. We see that the optimum value is slightly increased from 39 million to almost 40 million. The reason for this is the concave utility function where we want to reduce the probability of the large losses. □

Problem 5.7 Make a sketch of the utility function in Example 5.4. Discuss the utility when y approaches minus infinity and plus infinity. □

Problem 5.8 Use pRisk.xls to perform the calculations for the example in Example 5.4. Discuss the influence of the parameter b and discuss the effect of letting b approach zero. □

Extra effort in order to reduce penalties for default

We are a part of the project management for a large road development project and realise that it will be difficult to reach the completion date agreed upon. The following quantities describe the situation:

CD = 30 = Agreed completion date, in days from now

Y_1 $=T$ = Completion date, in days from now. $Y_1 \sim \text{PERT}(L, M, H)$.

PD = 10,000 Euro = Penalty for default, i.e the amount to pay each day the project is delayed (Y_1 >CD).

BO = 5,000 Euro= Bonus, i.e the amount BO is paid extra for each day the project is completed before CD.

Y_2 = EE = Extra effort we invest in order to speed up the project.

L_0 = 25 = Lowest value of Y_1, if nothing extra is done.

M_0 = 35 = Most likely value ofY_1, if nothing extra is done.

H_0 = 60 = Highest value of Y_1,if nothing extra is done.

L_{Y2} $=L_0(0.5 + 0.5e^{(-Y_2/50000)})$ = Lowest value of Y_1, with extra effort Y_2.

M_{Y2} $=M_0(0.5 + 0.5e^{(-Y_2/50000)})$ = Most likely value of Y_1, with extra effort Y_2.

H_{Y2} $=H_0(0.5 + 0.5e^{(-Y_2/50000)})$ = Highest value of Y_1, with extra effort Y_2.

We will use a linear utility function, and we let the utility of one Euro be equal to one. The following utility function the applies:

$$u(y_1, y_2) = PD \cdot (CD - y_1) \cdot I_{\text{CD}<y_1} + BO \cdot (y_1 - CD) \cdot I_{\text{CD}>y_1} - y_2 \tag{5.9}$$

Also here we utilise the pRisk.xls program to calculate expected utility. The following statements are entered into the Excel sheet:

```
CD=30
L_0=25
M_0=30
H_0=60
PD=10000
BO=5000
Y_2= 10000
LY_2=L_0*(0.5+0.5*exp(-Y_2/50000))
MY_2=M_0*(0.5+0.5*exp(-Y_2/50000))
```

```
HY_2=H_0*(0.5+0.5*exp(-Y_2/50000))
Y_1=RndPERT(RAND(),LY_2, MY_2, HY_2)
```

We obtain the following values for expected utility:

EE=Y_2	0	10 000	20 000	30 000	40 000	50 000
E(u(Y_1,Y_2))	-44 000	-28 000	-19 000	- 15 000	-15 000	-18 000

The maximum expected utility is obtained by spending between 30 and 40 thousand Euros in order to reduce the risk of delays in the project.

Problem 5.9 Use the pRisk.xls program to perform the simulation as indicated above. Check the sensitivity in the results as a function of the number of simulation runs. Then find a more exact result for the optimum value of the extra effort. □

5.2.4 Decision trees

The use of decision trees is a fruitful approach when we are going to systemise a decision process where the decisions are made at different point of times. The main reason for postponing a decision is to follow the development of e.g. a project, and hence make the most appropriate decision when more information is available. The drawback is that postponing a decision could yield more costly solutions. Another drawback could be that it is no time to implement necessary measures in due time if we wait to take action.

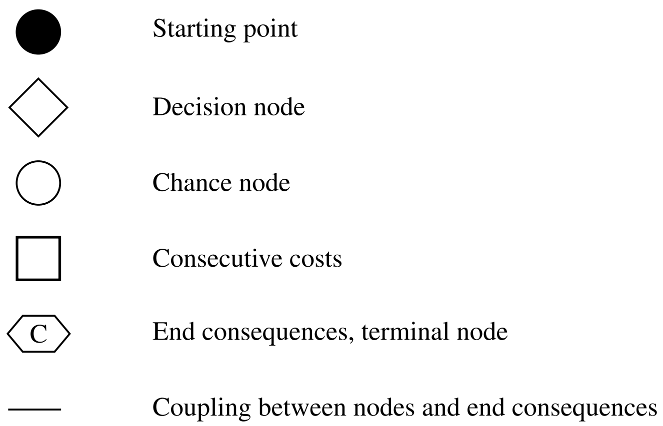


Figure 5.5: Symbols used in decision trees

Analysis of decision trees

The following notation is introduced:

```
Repeat for all end terminals
(*) Move to the node to the left, and bring with the EMV-value in the current node
  IF this is a chance node THEN
      Calculate EMV_{i,j} = p_{i,j}· EMV
      IF EMV has been calculated for all branches into this node THEN
          Calculate EMV_i = Σ_j EMV_{i,j}
          GoTo (*)
      ELSE
          GoTo next terminal node
      ENDIF
  ELSEIF this is a decision node THEN
      IF EMV has been calculated for all branches into this node THEN
          Let EMV_i = Min_j( EMV_{i,j})
          Optimum decision in DN_i is the branch with minimum EMV_{i,j}
          GoTo (*)
      ELSE
          GoTo next terminal node
      ENDIF
  ELSEIF this is a consecutive node THEN
      Add EMV of the consecutive node to EMV
      GoTo (*)
  ELSEIF this is the start node THEN
      We are done
  ENDIF
```

Figure 5.6: Algorithm for processing a decision tree

CN_i = Chance node i
$p_{i,j}$ = probability that chance node i results in outcome j.
DN_i = Decision node i.
CIN_j = Cost of intermediate node j
CTN_j = Cost related to terminal node j.
EMV = Expected Monetary Value .
$EMV_{i,j}$ = EMV for branch j into chance node i.
EMV_i = EMV for chance node i, or decision node i.

The algorithm for numeric calculating is shown in Figure 5.6.

Example 5.5
Construction Ltd. is the main contractor for a road tunnel project. During the work more water penetration than expected is discovered. Physically there are three alternatives to choose among:*i)* bursting an outlet drain which is very costly but a satisfactorily solution, *ii*) build a pumping station to pump away the water which is a cheaper solution but may not be adequate if there is very much water, and *iii*) carry out seal work which is even cheaper, but adequate only in case of very little water. The amount of water is uncertain at the time being. Below we discuss the decision process:

The first decision is now (DN_1), and at this decision node we have the following options:

A: Immediate start bursting work

B: Wait half a year until more information about the amount water is available

If we postpone the decision (B) we would have more information about the amount of water in half a year and a better decision could be made. Two outcomes are foreseen in half a year (CN_1):

C: It is obviously so much water that bursting the outlet drain is necessary

D: There is still uncertainty regarding the amount of water, and we have a new option in decision node DN_2:

E: Build a pumping station and hope that this is sufficient, or

F: Wait another half year to obtain even more information

If the pumping station is build at this time (E) this could result in the following outcomes (CN_2):

G: The pumping station was sufficient

H: The pumping station was not sufficient, and an outlet drain have to be bursted

If we wanted to postpone the decision (F) there are tree possible outcomes (CN_3):

I: Bursting the outlet drain is required

J: Sealing work is sufficient

K: A pumping station is sufficient

Table 5.1 shows the associate costs (the letter in parentheses corresponds to the alternative above).

Options	Cost now	Cost in half a year	Cost in one year
Outlet drainage bursting	50 mil. (A)	60 mil. (C)	70 mil. (I,H)
Pumping station		20 mil. (G)	25 mil. (K)
Seal work			10 mil. (J)

Table 5.1: Cost of the various options

At the moment we make the following probability assessments:

$P(C|CN_1) = 30\%$
$P(D|CN_1) = 70\%$
$P(G|CN_2) = 90\%$
$P(H|CN_2) = 10\%$

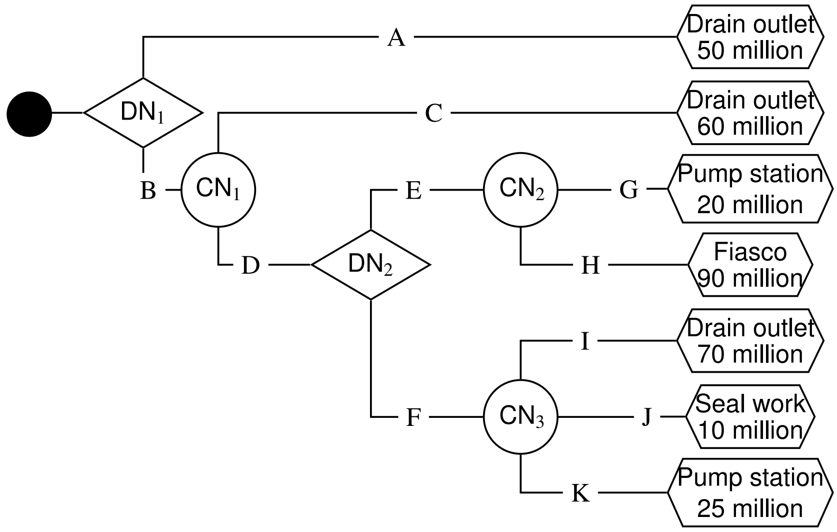


Figure 5.7: Decision tree for tunnel project

P(I|CN_3) = 10%
P(J|CN_3) = 40%
P(K|CN_3) = 50%

Note that in the example we have not used the symbol for consecutive costs. For the calculation we use the algorithm indicated in Figure 5.6. We start with the upper right terminal node, and "collect" the EMV = 50 mill. into the decision node to the left, e.g. DN_1. In this decision node we observe that not all branches (from the right) into node DN_1 have been processed, and we therefore need to go back to a new terminal node. We go back to the next upper terminal node and collect EMV = 60 mill. which is multiplied with the branch probability (30%) such that we get EMV = $0.3 \cdot 60$ mill. = 18 mill. into chance node CN_1. Here, the second branch into the chance node has not been processed and we again have to go back to the next non-processed terminal node. Here we collected EMV = 20 mil which is multiplied with 90% gives EMV = 18 mill. into chacne node CN_2. Similarily we get EMV = 90 mill. $\cdot$ 10% = 9 mill. for the second branch into chance noe CN_2. We may now complete the processing of chance node CN_2 by adding the EMV values entering the node from the right, yilding an EMV of 27 mill. This number now goes into decision node DN_2. Now the remaining end nodes are processed, and we get the EMV to collect from CN_3 equal to 23.5 which again will be the second EMV into decision node DN_2. In decision node DN_2 we shall choose the branch having the lowest EMV value, i.e. branch F with an EMV of 23.5. In decision node DN_2 it is most beneficial to postpone the decision for another half year. We complete the tree similarly, and find that in decision node DN_1 the optimal decision is to postpone any physical activity. We remain then with an EMV equal to 34.45. The number from these calculations are shown in Figure 5.8. Also note that we have not taken the discounting aspects into account, something that also would have

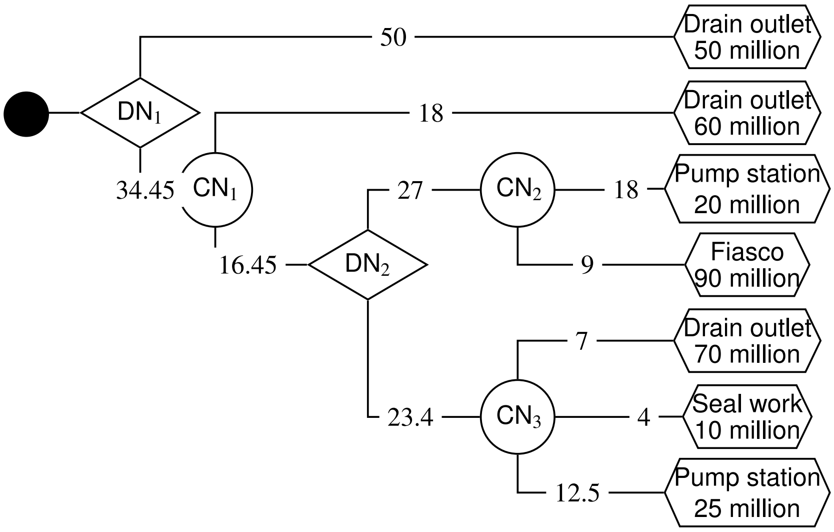


Figure 5.8: Decision tree for tunnel project with calculations

been an argument for postponing the decision. □

Problem 5.10 An oil company has the rights for a given oil field, and have the options between:

- Drill a well (D)
- Sell our rights (S)

The decision will depend on the amount of oil that might reside in the field. There are two options:

- Profitable oil pool (P)
- Non profitable oil pool (NP)

Before the final decision is made, our oil company could conduct an expensive seismic investigation which might give information regarding the probability that the field contains a profitable oil pool. The result from such an investigation will be one of the following statements:

- No structure (NS)
- Open structure (OS)
- Closed structure (CS)

In this problem you should establish a decision tree for the situation. The decision node following the start node should be:

- Perform a seismic investigation (SEI)
- Do not perform a seismic investigation (NINV)

The cost involved in the decision tree is as follows:

Cost of seismic inveestigation:	10
Cost of drilling a well:	100
Net profit (**after** drilling) if oil:	700
Selling price without seismic investigation:	30
Selling price **with** seismic investigation: - No structure - Open structure - No structure	 0 50 200

The probability that shall go into the decision tree is as follows:

Probabilities in relation to a seismic investigation (e.g. based on experience figures from geologists):

No structure:	0.60
Open structure:	0.30
Closed structure:	0.10

The probabilities for finding a profitable oil pool, given the result of the seismic investigation:

No structure:	0.10
Open structure:	0.25
Closed structure:	0.70

The probability for finding oil when no seismic investigation has been performed: 0.20. Find the optimal decision in each decision node, and formulate the conclusions from your analysis. □

Chapter 6

Life cycle cost and life cycle profit

6.1 Introduction

In this chapter we will give a short introduction to life cycle cost (LCC) modelling and anaylysis in connection with project management. The term LCC is defined in IEC 60300: "LCC is the cumulative cost of a product over its life cylce". The LCC concept was first introduced in the US Army and the idea was to establish the cost of development, production and use (operation and maintenance) of military equipment. In the original use the revenues was not included in the modelling. However, in order to get a complete picture we will usually also include the possible profit of a new system or product. Hence the term 'Life Cycle Profit' has been introduced. Kawacuchi and Rausand [8] suggest a process for LCC analysis comprising the following steps:

1. Problem definition
2. Cost element definition
3. System modelling
4. Data collection
5. Cost profile development
6. Evaluation
7. Reporting

In this presentation we will focus on the cost modelling aspects, i.e. mainly step 3 in the above procedure.

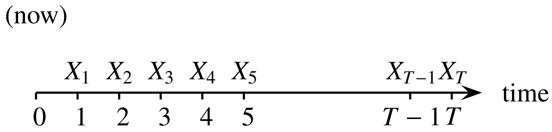


Figure 6.1: Visualisation of the cach flow, X_t

6.2 Net present value calculation

The formulas for LCC calculation is based on standard formulas used in net present value (NPV) calculations. In the following we will sumarise the most frequent used formulas. The basic idea in NPV calculation is that money received in the future will be less valued than the same amount of money today. To treat this formally all future amounts are discounted to the present time, i.e. present values. We will only consider discrete time, i.e. all amounts will occur at the end of each year, or now (beginning of year one). The cash flow is illustrated in Figure 6.1.

The net present value of an amount X_t that occurs at the end of year t is:

$$\text{NPV} = X_t(1+r)^{-t} \tag{6.1}$$

where r is the discount rate. Similarly, we find the net present value of a cash flow $X_0, X_1 \ldots, X_T$:

$$\text{NPV} = \sum_{t=0}^{T} X_t(1+r)^{-t} \tag{6.2}$$

where X_0 represents in or outgoing cash now, and T is the number of years to consider.

Sometimes we want to establish the net present value of a constant yearly (nominal) amount X_A, i.e. the same amount each year. By utilising the formula for the sum of a geometric serials, $\sum_{i=1}^{n} q^i = q(1-q^n)/(1-q)$ we obtain:

$$\text{NPV} = \left[\frac{1-(1+r)^{-T}}{r}\right] X_A \tag{6.3}$$

Note that NPV approaches X_A/r as T approaches infinity.

Now, consider a situation with a fixed increasing yearly value, were the first in or outgoing amount is $X_{A,v}$ (at the end of the first year), and where the amount is increasing by a factor $(1+v)$ each year. The net present value for T years is then found to be:

$$\text{NPV} = \left[\frac{1-\left(\frac{1+v}{1+r}\right)^T}{r-v}\right] X_{A,v} \tag{6.4}$$

IF $r = v$ in Equation (6.4) we use NPV $= X_{A,v}T/(1+r)$ obtained by l'Hopitals rule.

The expression in Equation (6.3) assumes that the amount $X_{A,v}$ occurs every year. In some situations we want to consider an amount $X_{A,v}$ which occurs every k year, where $k > 1$. The net present value is now given by (assuming first amount now ($t = 0$)):

$$\text{NPV} = \sum_{i=0}^{\infty} X_A(1+r)^{ki} = \frac{X_A}{1-(1+r)^{-k}} \tag{6.5}$$

If the first amount occurs at the end of year l we obtain:

$$\text{NPV} = \frac{X_A(1+r)^{-l}}{1-(1+r)^{-k}} \tag{6.6}$$

6.2.1 Trend modelling

When modelling trend it is important to find a simple mathematical expression for the time development. Further note that the change in the yearly amount is due to at least the following factors:

- The monetary value increases due to general conditions, such as inflation.
- The monetary value increases due to increased operating costs, e.g. physical deterioration and hence more maintenance is required.

Increased operating costs due to deterioration could usually be reset by a renewal of the system we are considering, whereas external conditions like inflation is not affected by e.g. a system renewal. In the modellling we will assume a fixed inflation rate, even if we in a more advanced model also could let the inflation rate vary. This inflation rate will be denoted v, and we could use Equation 6.4 to calculate the net present value of a amount that changes due to inflation. When we want to model increased operating cost due to deterioration, we need to introduce a local age parameter. We will let a denote the age of the system, or the age of the system since the last system renewal. When we consider degradation, we introduce the degradation rate d where we assume that the yearly increase due to deterioration equals $(1 + d)$. This corresponds to an exponential growth which very often is found realistic if we have degradation mechanisms that drive the costs. Now, let c_0 be the yearly cost of operation, maintenance etc now (i.e. at time $t = 0$). We then have the yearly cost in year t (occurring at the end of year t):

$$c_t = c_0(1+d)^t \tag{6.7}$$

In order to obtain the degradation rate d we usually need data about the costs as a function of time. A very simple approach if we know that $c(t)$ has increased by a growth factor (GF) during a period of T years. We then have:

$$d = e^{\ln(\text{GF})/T} - 1 \tag{6.8}$$

6.2.2 Example areas of LCC calculations

In the following we give examples of areas where LCC analysis and calculation could be used. We differentiate between situations were decisions are related to project execution, and the progression of one project, or a portfolio of projects, and the situation where we consider which project are profitable, or how the profitability could be maximised. Examples related to project execution:

- Invest in equipment to increase efficiency in project execution, e.g. a new excavator.

- Choice between construction method *A* and *B*.
- Outsourcing of truck-maintenance.
- Lease equipment rather than by our selves.

Examples related to project profitability:

- Development of one or more oil fields.
- Construction of a new passing loop.
- Renewal of ballast in a railway track.
- Point wise refill of ballast in order to postpone the need for a full renewal (ballast cleaning).
- Grinding of rails.
- Invest in a new production line.

There are several aspects to consider when conducting an LCC analysis, for example:

- Visualise the cost picture, enabling the possibility to work actively with eliminating the main cost drivers, or the effect of these.
- Use the LCC model as a decision support when making decision about the profitability of projects or measures, and when to conduct or implement these.
- Use the LCC model as a basis for contractual follow-up, e.g. LCC contracts.

Example 6.1 We will consider a railway system where the quality of the ballast has deteriorated during the last years, and in order to compensate for this it is proposed to do a point wise replacement of the ballast on the line. The age of the ballast is 35 years, and without this point wise refill of ballast it is expected that a full renewal (ballast cleaning) is necessary within five years. If we conduct the project we could postpone the ballast cleaning with another five year. The length of the line we are considering is 10 km. The quantities to include in the LCC model is as follows:

RC = 2.5 million Euro = Renewal cost = 250 Euro per meter for ballast cleaning.

IC = 400,000 Euro= Improvement cost, e.g. cost of point wise ballast refill.

LT = 40 years = Life length of ballast = period between ballast cleaning.

a = ballast age, i.e. effective age relative to the implemented measures. Without point wise refill of ballast $a = 35$ years, and with point wise refill of ballast $a = 30$ year. For a track that has just being renewed $a = 0$.

c_0 = 25,000 Euro = yearly cost of maintenance and operation of the track, for a new track, i.e. just being renewed.

c_{40} = 250,000 Euro = yearly cost of maintenance and operation of the track, for a track that has reached it's service life, e.g. 40 years.

$d = e^{\ln(250000/2500)/40} - 1 = 0.05925$

$c_t = c_0(1+d)^{t+a} = 25000(1+0.05925)^{t+a}$ = total maintenance and operation cost in year t (from now), and a is the effective age of the track.

r = 6% = interest rent.

We start by calculating the various LCC-terms (in million Euros) if the improvement project (point wise refill of ballast) is not executed. The total renewal cost if found by Equation (6.6):

$$\mathrm{LCC}_{\mathrm{RC}} = \frac{\mathrm{RC}(1+r)^{-5}}{1-(1-r)^{-40}} = 2.069$$

The variable cost the next five years (up to the next renewal) is found from Equation (6.4)

$$\mathrm{LCC}_{\mathrm{VC},1} = \left[\frac{1-\left(\frac{1+d}{1+r}\right)^{5}}{r-d}\right] c_0(1+d)^{35} = 0.883$$

After the renewal in five year the variable costs will be reset to v_0, and then start increasing again. The net present value in one cycle is:

$$\mathrm{LCC}_{\mathrm{VC},0} = \left[\frac{1-\left(\frac{1+d}{1+r}\right)^{40}}{r-d}\right] c_0(1+d) = 0.986$$

The amount $\mathrm{LCC}_{\mathrm{VC},0}$ will then be repeated every 40 year, and the first time will be in five years:

$$\mathrm{LCC}_{\mathrm{VC},\infty} = \frac{\mathrm{LCC}_{\mathrm{VC},0}(1+r)^{-5}}{1-(1+r)^{-40}} = 0.816$$

Finally we have the total contribution from variable costs:

$$\mathrm{LCC}_{\mathrm{VC}} = \mathrm{LCC}_{\mathrm{VC},1} + \mathrm{LCC}_{\mathrm{VC},\infty} = 1.699$$

If we execute the improvement project, the calculations are similar. We start with the total renewal cost (first renewal after 10 years):

$$\mathrm{LCC}_{\mathrm{RC}} = \frac{\mathrm{RC}(1+r)^{-10}}{1-(1+r)^{-40}} = 1.546$$

The variable cost the next ten years (up to the next renewal) noting that the effective age after the improvement project is $a = 30$:

$$\mathrm{LCC}_{\mathrm{VC},1} = \left[\frac{1-\left(\frac{1+d}{1+r}\right)^{10}}{r-d}\right] c_0(1+d)^{3}0 = 1.322$$

After the renewal in ten year the variable costs will be reset to v_0, and then start increasing again. The net present value in one cycle, $\mathrm{LCC}_{\mathrm{VC},0}$, is the same as without the improvement project, but the first cycle will start in ten years:

$$\mathrm{LCC}_{\mathrm{VC},\infty} = \frac{\mathrm{LCC}_{\mathrm{VC},0}(1+r)^{-10}}{1-(1+r)^{-40}} = 0.610$$

Finally we have the total contribution from variable costs:

$$\mathrm{LCC}_{\mathrm{VC}} = \mathrm{LCC}_{\mathrm{VC},1} + \mathrm{LCC}_{\mathrm{VC},\infty} = 1.932$$

In this last situation we also need to include the investment cost:

$$\mathrm{LCC}_{\mathrm{IC}} = 0.4$$

Summing up all LCC contributions we find that implementing the improvement project gives a total LCC of 3.878 million versus not implementing the project gives a total cost of 3.768. Thus the improvement project is not profitable. □

Chapter 7

Parameter estimation

7.1 Introduction

In this chapter we will briefly describe principles for parameter estimation. A parameter in this context is a quantity in the risk analysis for which we assign numerical values. There are two principles for establishing numerical values (parameter estimates):

- Statistical analysis of historical data
- Use of expert judgements

If we have access to relevant data we will usually use these data to estimate the parameters. Often we have little relevant data, and we then have to rely on expert judgements. In some situations we combine historical data with expert judgements by use of Bayesian methods.

7.2 The MLE principle

The basic idea behind the Maximum Likelihood Estimation (MLE) principle is to choose the numerical values of the parameters that are the most likely ones in light of the data. The procedure goes as follows:

- Assume that we know the probability density function of the observations for which we have data. Let this distribution be denoted $f(x;\boldsymbol{\theta})$.
- The involved parameters, are unknown, and are generally denoted $\boldsymbol{\theta}$.
- We have n independent observations (data points) that we denote $X_1, X_2, \dots X_n$. When we refer to the actual numerical values we have observed, we use the notation $x_1, x_2, \dots x_n$.

The MLE principle now tells us to estimate $\boldsymbol{\theta}$ by the value which is most likely given the observed data. To define "likelihood" we use the probability density function. The simultaneous probability density for X_1, X_2, $\ldots X_n$ is given by:

$$f(x_1;\boldsymbol{\theta})f(x_2;\boldsymbol{\theta})\ldots f(x_n;\boldsymbol{\theta}) = \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta}) \tag{7.1}$$

This density express how likely a given combination of the x-values are, given the value of $\boldsymbol{\theta}$. However, in our situation the x-values are given, whereas $\boldsymbol{\theta}$ is unknown. We therefore interchange the arguments, and consider the expression as a function of $\boldsymbol{\theta}$:

$$L(\boldsymbol{\theta}; x_1, x_1 \ldots x_n) = \prod_{i=1}^{n} f(x_i;\boldsymbol{\theta}) \tag{7.2}$$

where $L(\boldsymbol{\theta}; x_1, x_1 \ldots x_n)$ in equation (7.2) denotes the likelihood function. The MLE principle will now be formulated as to choose the $\boldsymbol{\theta}$-value that maximizes the likelihood function. To denote the MLE *estimator* we write a "'hat" over $\boldsymbol{\theta}$, $\hat{\boldsymbol{\theta}}$. Generally, $\boldsymbol{\theta}$ will be a function of the observations:

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(X_1, X_2, \ldots X_n) \tag{7.3}$$

When we insert numerical values for the x's we denote the result as the parameter *estimate*.

Example 7.1 Estimation in the exponential distribution
We consider the situation where we have observed n failure times, and we will estimate the failure rate, λ, under the assumption of exponentially distributed failure times.The observed failure times are denoted t_1, t_2, …, t_n. Equation (7.2) gives:

$$L(\lambda; t_1, t_2, \ldots, t_n) = \prod\nolimits_{i=1}^{n} \lambda e^{-\lambda t_i}$$

Note that the parameter is denoted λ, whereas we generally use θ**.** Further we denote the observations with t because we here have failure *times*. The probability density function in the exponential distribution is given by $f(t) = \lambda \mathrm{e}^{-\lambda t}$. A common "trick" when maximising the likelihood function is to take the logarithm. Because the logarithm (ln) function is monotonically increasing, ln L will also be maximised for the same value as for which L is maximised. We could then find:

$$l(\lambda; t_1, t_2, \ldots, t_n) = \ln L(\lambda; t_1, t_2, \ldots, t_n) = n \ln \lambda - \sum\nolimits_{i=1}^{n} \lambda t_i$$

By taking the derivative wrt λ and set this expression to zero, we easily obtain:

$$\hat{\lambda} = n \Big/ \sum\nolimits_{i=1}^{n} t_i$$

□

Problem 7.1 Find the MLE for μ and σ in the normal distribution. □

7.3 Method of moments – PERT distribution

The maximum likelihood principle is not numerically stable for the PERT distribution. We will therefore apply another principle, i.e., the methods of moments. The method of moments is a method of estimation of population parameters by equating sample moments with unobservable population moments and then solving those equations for the quantities to be estimated.

In the PERT distribution we have the following moments:

$$\mathrm{E}(X) = \frac{L + 4M + H}{6} \tag{7.4}$$

$$\mathrm{Var}(X) = \frac{(\mathrm{E}(X) - L)(H - \mathrm{E}(X))}{7} \tag{7.5}$$

From (7.4) we easily obtain $M = \frac{1}{4}(6\mathrm{E}(X) - L - H)$. Now let $\bar{x} = \sum_i x_i$ denote the sample mean (first order moment), and we may estimate M by:

$$\hat{M} = \frac{1}{4}(6\bar{x} - \hat{L} - \hat{H}) \tag{7.6}$$

The challenge now is to find $\hat{L}$ and $\hat{H}$. By rearranging equation (7.5) and inserting $\bar{x}$ for $\mathrm{E}(X)$ and the sample variance $S^2 = \frac{1}{n-1}\sum_i (x_i - \bar{x})^2$ for $\mathrm{Var}(X)$ we have:

$$7S^2 = (\bar{x} - L)(H - \bar{x}) \tag{7.7}$$

where L and H are the two unknowns to find by one equation. To overcome the problem of underdetermination it seems reasonable to require $(\bar{x}-L)/(H-\bar{x}) = (\bar{x}-x_{\mathrm{Min}})/(x_{\mathrm{Max}}-\bar{x})$ where x_{Min} and x_{Max} are the sample minimum and maximum respectively. Hence we have:

$$7S^2 = (H - \bar{x})^2 \frac{\bar{x} - x_{\mathrm{Min}}}{x_{\mathrm{Max}} - \bar{x}} \tag{7.8}$$

yielding

$$\hat{H} = \bar{x} + S\sqrt{7\frac{x_{\mathrm{Max}} - \bar{x}}{\bar{x} - x_{\mathrm{Min}}}} \tag{7.9}$$

and

$$\hat{L} = \bar{x} - \frac{(\hat{H} - \bar{x})(\bar{x} - x_{\mathrm{Min}})}{x_{\mathrm{Max}} - \bar{x}} \tag{7.10}$$

thus, the final estimates in the PERT distribution are given by equations (7.6), (7.9) and (7.10).

Problem 7.2 Assume that we have observed the following durations for a typical activity in projects: 9.3, 10.5, 9.4, 9.0, 9.4, 8.6, 10.1, 10.7, 12.0, 10.6, 9.8, 13.1, 12.0, 8.6 and 10.9. Estimate the parameters if you assume that the durations are PERT distributed. Hint: Use the Average() and STDEV() functions in MS Excel to find $\bar{x}$ and S. □

7.4 The LS principle

The least squares (LS) principle for estimation is used when we have observations that do not come from the same distribution, but we know the expectation of each variable as a function of a set of parameters $\boldsymbol{\theta}$, and a set of explanatory variables. Previously we denoted the observations by the letter 'X', but we will now change the notation to let 'Y' denote the observations, whereas we reserve the letter 'X' for explanatory variables. We now let $\phi_i(\boldsymbol{\theta})$ denote the expectation of Y_i (the i'th observation), where the functions ϕ_i are all known, but the parameter vector $\boldsymbol{\theta}$ is unknown and shall be estimated. The LS principle now states that we may estimate $\boldsymbol{\theta}$ by the value that minimises the square sum of the deviations between the observed and expected values, i.e.:

$$Q(\boldsymbol{\theta}) = \sum_{i=1}^{n} [y_i - \phi_i(\boldsymbol{\theta})]^2 \tag{7.11}$$

Equation (7.11) is the starting point for estimating the parameters in so-called regression models. The most simple formula is given by:

$$E(Y_i) = \beta_0 + \beta_1 x_i \tag{7.12}$$

In this model x is denoted the *independent* variable, whereas Y is denoted the *dependent* variable because it depends on the independent variable, x.

Problem 7.3 Prove that the estimators for β_0 and β_1 in equation (7.12) is given by: $\hat{\beta}_1 = \frac{\sum_i (x_i - \bar{x}) y_i}{\sum_i (x_i - \bar{x})^2}$ and $\hat{\beta}_0 = \bar{y} - \hat{\beta}_1 \bar{x}$.

□

The model in equation (7.12) could be extended to cover more independent variables. These are denoted regression variables , or explanatory variables. To extend the model we introduce an extra index for each x. We write x_{ij}, where index i denotes the i'th data point, whereas index j denotes the j'th explanatory variable. The model then reads:

$$\mathrm{E}(Y_i) = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_r x_{i,r} \tag{7.13}$$

To obtain the LS estimators in this situation, we introduce matrix notation. Let $\mathbf{y} = [y_1, y_2, \ldots, y_n]^{\mathrm{T}}$ be a column vector containing the dependent variables, and let $\mathbf{X}$ be the *design matrix* given by:

$$\mathbf{X} = \begin{bmatrix} 1 & x_{11} & \ldots\ldots & x_{1r} \\ 1 & x_{21} & & x_{1r} \\ : & & x_{ij} & \\ 1 & x_{i1} & \ldots\ldots. & x_{nr} \end{bmatrix} \tag{7.14}$$

It could be shown that the LS estimator for $\boldsymbol{\beta} = [\beta_0, \beta_1, \beta_2, \cdots, \beta_r]^{\mathrm{T}}$ is given as the solution of the following matrix equation:

$$\mathbf{X}^{\mathrm{T}}\mathbf{y} = \mathbf{X}^{\mathrm{T}}\mathbf{X}\boldsymbol{\beta} \tag{7.15}$$

If the design matrix has full rank, $\mathbf{X}^\mathrm{T}\mathbf{X}$ will be non-singular, and the solution is given by:

$$\hat{\boldsymbol{\beta}} = (\mathbf{X}^\mathrm{T}\mathbf{X})^{-1}\mathbf{X}^\mathrm{T}\mathbf{y} \tag{7.16}$$

If one has access to a tool for matrix calculus, we easily obtain the LS estimates. We could also use commercial available statistical programs, or the "analysis" module of MS Excel.

Example 7.2 Estimation of the effects of regression variables
We will consider a situation where we have observed the duration of construction the foundation wall of houses. The different values are shown in the Y-column below. The variable x_1 denotes the base in square meters, whereas x_2 is an indicator of ground frost. A value is given as 1 if there is ground frost, 0 otherwise. We have also introduced the variable x_3 that denotes the walking distance from the workmen's hut to the building site:

Y	x_1	x_2	x_3
8.4	100	1	100
7.8	150	1	50
11.4	250	1	50
6.1	80	0	75
6.1	100	0	200
8.3	90	1	30
7.5	180	0	25
7.2	200	0	50
6.0	110	0	75

From MS Excel we obtain the following parameters: $\hat{\beta}_0 = 4.211$, $\hat{\beta}_1 = 0.0167$, $\hat{\beta}_2 = 2.196$, and $\hat{\beta}_3 = 0.0011$

□

Problem 7.4 Use MS Excel to verify the above results. □

Note that we in equation (7.13) have written the *expected* value of Y_i. Generally we write:

$$Y_i = \beta_0 + \beta_1 x_{i,1} + \beta_2 x_{i,2} + \cdots + \beta_r x_{i,r} + \varepsilon_i \tag{7.17}$$

where ε_i is an error-term. Very often we assume ε_i to be normally distributed, but we might also assume that ε_i is PERT distributed. To estimate the parameters in an underlying PERT distribution we calculate the predicted values:

$$\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i,1} + \hat{\beta}_2 x_{i,2} + \cdots + \hat{\beta}_r x_{i,r} \tag{7.18}$$

then we estimate the error-terms by the residuals:

$$\hat{\varepsilon}_i = y_i - \hat{y}_i \tag{7.19}$$

The residuals $\hat{\varepsilon}_i$ may now be used as input in the method of moments to estimate PERT parameters. Below se summarize the procedure to get L, M and H values for an activity, cost element etc. in a specific project.

1. Estimate regression parameters from data from similar projects: $\hat{\boldsymbol{\beta}} = (\mathbf{X}^{\mathrm{T}}\mathbf{X})^{-1}\mathbf{X}^{\mathrm{T}}\mathbf{y}$

2. Calculate the predicted values: $\hat{y}_i = \hat{\beta}_0 + \hat{\beta}_1 x_{i,1} + \hat{\beta}_2 x_{i,2} + \cdots + \hat{\beta}_r x_{i,r}$

3. Estimate the error-terms by the residuals: $\hat{\varepsilon}_i = y_i - \hat{y}_i$

4. Use the estimates $\hat{\varepsilon}_i$ as basis for estimation of L, M and H, i.e., find $\hat{L}$, $\hat{M}$ and $\hat{H}$ by the method of moments

5. For the new activity, cost element etc., find the corresponding x-vector, and denote it $x = [x_1, x_2, \ldots, x_r]$

6. Find the prediction of the new observation by $y_0 = \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 + \cdots + \hat{\beta}_r x_r$

7. The PERT parameters to use in the analysis are given by $L = \hat{L} + y_0$, $M = \hat{M} + y_0$ and $H = \hat{H} + y_0$

Problem 7.5 Calculate the residuals in equation (7.19) with the data in Example **7.2** and estimate the parameters by assuming the residuals are PERT distributed. Find the probability that the duration for observation number 5 is shorter than the observed value of 6.1. From the observations it seems that duration number 3 is rather long. Conclude on this by applying the regression model. □

7.5 Bayesian meothds

In Bayesian estimation procedures we utilise prior information about the reliability parameters. The procedure could briefly be described as follows:

1. Specify a *prior* uncertainty distribution of the reliability parameter, $\pi(\boldsymbol{\theta})$.

2. Structure reliability data information into a likelihood function, $L(\boldsymbol{\theta}; \mathbf{x})$, see equation (7.2).

3. Calculate the *posterior* uncertainty distribution of the reliability parameter vector, $\pi(\boldsymbol{\theta}|\mathbf{x})$. The posterior distribution is found by $\pi(\boldsymbol{\theta}|\mathbf{x}) \propto L(\boldsymbol{\theta}; \mathbf{x})\pi(\boldsymbol{\theta})$, and the proportionality constant is found by requiring the posterior to integrate to one.

4. The Bayes estimate for the reliability parameter is given by the posterior mean, which in principle could be found by integration.

Example 7.3 Exponential distribution
In the following we give an example showing the main elements of the procedure. In the example we will estimate the failure rate in the constant failure rate situation. Assume that we express our prior believe[1] about the failure rate λ of a certain component (gas detector used on an oil and gas platform), in terms of the mean value $E = 0.7 \cdot 10^{-6}$

[1] This could be based on statements from experts, see Øien et.al (1998), or by analysis of similar components (empirical Bayesian analysis).

(failures / hour), and the standard deviation $S = 0.3 \cdot 10^{-6}$. For mathematical convenience, it is common to choose a gamma distribution[2] with parameters *alpha* and ξ for the prior distribution. The expected value (E) and the variance (V) in the gamma distribution are given by α/ξ and α/ξ^2 respectively, and we obtain the following expressions for α and ξ:

$$\xi = E/V = E/S^2 = (0.7 \cdot 10^{-6})/(0.3 \cdot 10^{-6})^2 = 7.78 \cdot 10^6$$
$$\alpha = \xi E = (7.78 \cdot 10^6) \cdot (0.7 \cdot 10^{-6}) = 5.44$$

To establish the likelihood function, we look at the data. In this example we assume that we have observed identical units for a total time in service, t, equal to 525 600 hours (e.g. 60 detector years). In this period we have observed $n = 1$ failure. If we assume exponentially distributed failure times, we know that the number of failures in a period of length $t, N(t)$, is Poisson distributed with parameter $\lambda \cdot t$. The probability of observing n failures is thus given by:

$$L(\lambda; n, t) = \Pr(N(t) = n) \propto \lambda^n e^{-\lambda \cdot t} \tag{7.20}$$

and we have an expression for the likelihood function $L(\lambda;n, t)$.

The posterior distribution is found by multiplying the prior distribution with the likelihood function:

$$\pi(\lambda|n) \propto L(\lambda; n, t) \cdot \pi(\lambda) \propto \lambda^n e^{-\lambda \cdot t} \cdot \lambda^{\alpha-1} e^{-\xi\lambda} \propto \lambda^{(\alpha+n)-1} e^{-(\xi+t)\lambda} \tag{7.21}$$

and we recognize the posterior distribution as a gamma distribution with new parameters $\alpha' = \alpha + n$, and $\xi' = \xi + t$. The Bayes estimate is given by the mean in this distribution:

$$\hat{\lambda} = \frac{\alpha + n}{\xi + t} = \frac{5.44 + 1}{7.78 \cdot 10^6 + 525600} = 0.78 \cdot 10^{-6}$$

We note that the maximum likelihood estimator gives a much higher failure rate estimate ($1.9 \cdot 10^{-6}$), but the "weighing procedure" favours the prior distribution in our example. Generally we could interpret α and ξ here as "number of failures" and "time in service" respectively for the "prior information". □

[2] $\pi(\lambda) \propto \lambda^{\alpha-1} e^{-\xi\lambda}$ for the gamma distribution.

Chapter 8

Expert judgments

Note that this chapter is widely based on: Øien, K. & Hokstad, P.1998 "Handbook for performing Expert Judgment". SINTEF report STF38 A98419 (ISBN 82-14-00449-7). Whereas Øien & Hokstad have a general approach, we will here focus on estimation where the objective is to assess two or more parameters describing the uncertainty distribution of durations or costs.

8.1 Introduction

When statistical data do not exist or are not available, the alternative is to obtain such information from experts/resource persons. To handle this in the best possible way, this should be carried out as a structured and systematic process, both during planning, elicitation and calculation, as proposed in this handbook. Such a structured collection of information is what we call "expert judgments". Thus, expert judgments are carried out to provide necessary input data for our analyses.

ESA [17] gives the following definition of *expert judgment data:*Expert judgment!Data

"Expert Judgment data are estimates of unknown values about a system made by specialists who have system-related knowledge."

8.1.1 Purpose

The purpose of this chapter is primarily to give a simple and complete "recipe" of how expert judgments may be carried out. Note that the referred handbook by Øyen and Hokstad gives a significant more detailed picture than this chapter. At the end of the chapter combining expert judgment findings with statistical data is also discussed.

8.1.2 Extent

Figure 8.1 shows tasks that are covered for performing expert judgments. This chapter covers all steps of all 3 phases.

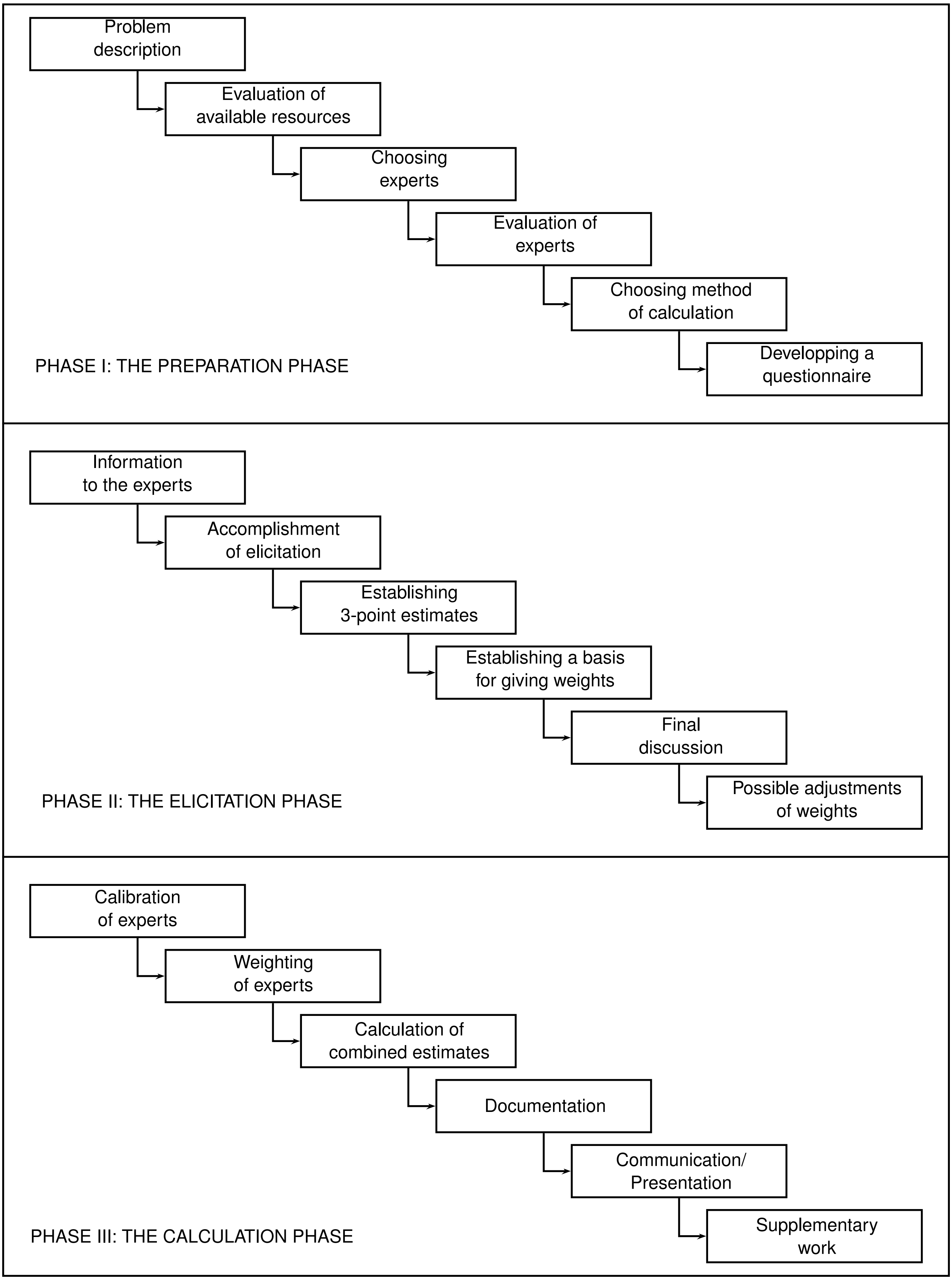


Figure 8.1: The steps in an expert judgment process

8.1.3 Use

It has been an overall objective to make this chapter flexible, that is, also to be applicable for expert judgments carried out with little resources, and thereby in a simple way. In addition, there are guidelines of how to perform more comprehensive expert judgments.

Therefore, it is *not* the intention that all the procedure items are to be carried out every time. Which items are to be carried out depend on which resources are available, the problem to solve, and the choices made during the work/process, (among others: choosing method of calculation). All items should therefore be looked into, in order to evaluate whether they are relevant. The short version in Section 3 gives a suggestion for what to include for very simple expert judgments.

This chapter represents an alternative to an unstructured and undocumented "engineering" judgments.

8.2 General theory

8.2.1 History

Speculations, brainstorming and guessing made by experts and used as basis in structured decision processes, is of relatively new date. It all started with the establishment of RAND Corporation in the USA after the second World War. At first, two methodologies were dominating. These were the scenario method and the Delphi method. Both were developed at RAND.

Scenario analyses

Herman Kahn developed this method as a kind of system analyses method, where hypothetical sequences of events are constructed for the purpose of focusing on cause processes and decision points. This gives the answers to two kinds of questions:

1. Precisely how might some hypothetical situation come about, step by step?

2. What alternatives exist, for each actor, at each step, for preventing, diverting, or facilitating the process?

This type of scenario analysis must not be confused with accident scenarios, and are only used to examine the main trends, which are extrapolated into the future. This is done without evaluating or using the probabilities of the scenarios to occur.

When the trends are extrapolated into the future, any theoretical or experience based knowledge that may influence the extrapolation is considered.

The Delphi method

This method is also developed at RAND [30] and is the most well-known method to obtain and treat the experts' opinions/judgments.

The Delphi method can be summarised into 8 steps:

1. An "observation team" (the analysts) defines the purposes and choose the respondents (experts). Normally, the respondents are anonymous to each other, and the answers are anonymous.

2. The observation team prepares a temporary questionnaire, which is sent to the respondents for comments.

3. The answers are reviewed, and a final questionnaire is established.

4. The respondents answer the questionnaires.

5. The observation team analyses the answers and calculates median values and the interquartile range (the 25% and the 75% estimates).

6. The results are returned to the respondents, who are asked whether they want to adjust their answers. Those who still are outside the interquartile range, are asked to give arguments for their prediction.

7. The revised predictions are processed in the same way as the first responses, and arguments for "outliers" are summarised. This information is sent back to the respondents, and the whole process is iterated (3 - 4 times).

8. The median values on the final round are taken as the best predictions. Generally, the spread in the last round is smaller than in the first round, which is taken to indicate a degree of consensus.

This method was very popular in the 1960's and 1970's, but some later evaluation studies which have seen carefully and critically on the Delphi method, have concluded that it violates methodological rules for common experimental science. Comparisons with other methods have also shown that the Delphi method was the poorest.

One major criticism is directed to the "reward" given to the expert for changing his estimates towards the median value through group interaction. It has been shown that this does not increase the relative frequency of correct estimates. The Delphi method seems to have lost its popularity.

8.2.2 Which types of evaluations/estimates do we consider as expert judgments?

In practice you can be an expert in any field or any topic, and in this respect one could regard any information or knowledge about this as expertise, and treat it as "expert judgments". However, in this chapter we will limit ourselves to a specific type of expert judgment, as we ask the experts to give their knowledge/experience on a subject.

There is no clear definition in the literature about what type of information that may be seen as expert judgments. However, in most of the literature, expert judgments are performed to "characterise" the collection of quantitative data in lack of statistical data, (e.g. subjective probabilities). Most likely duration of an activity is an example of such data.

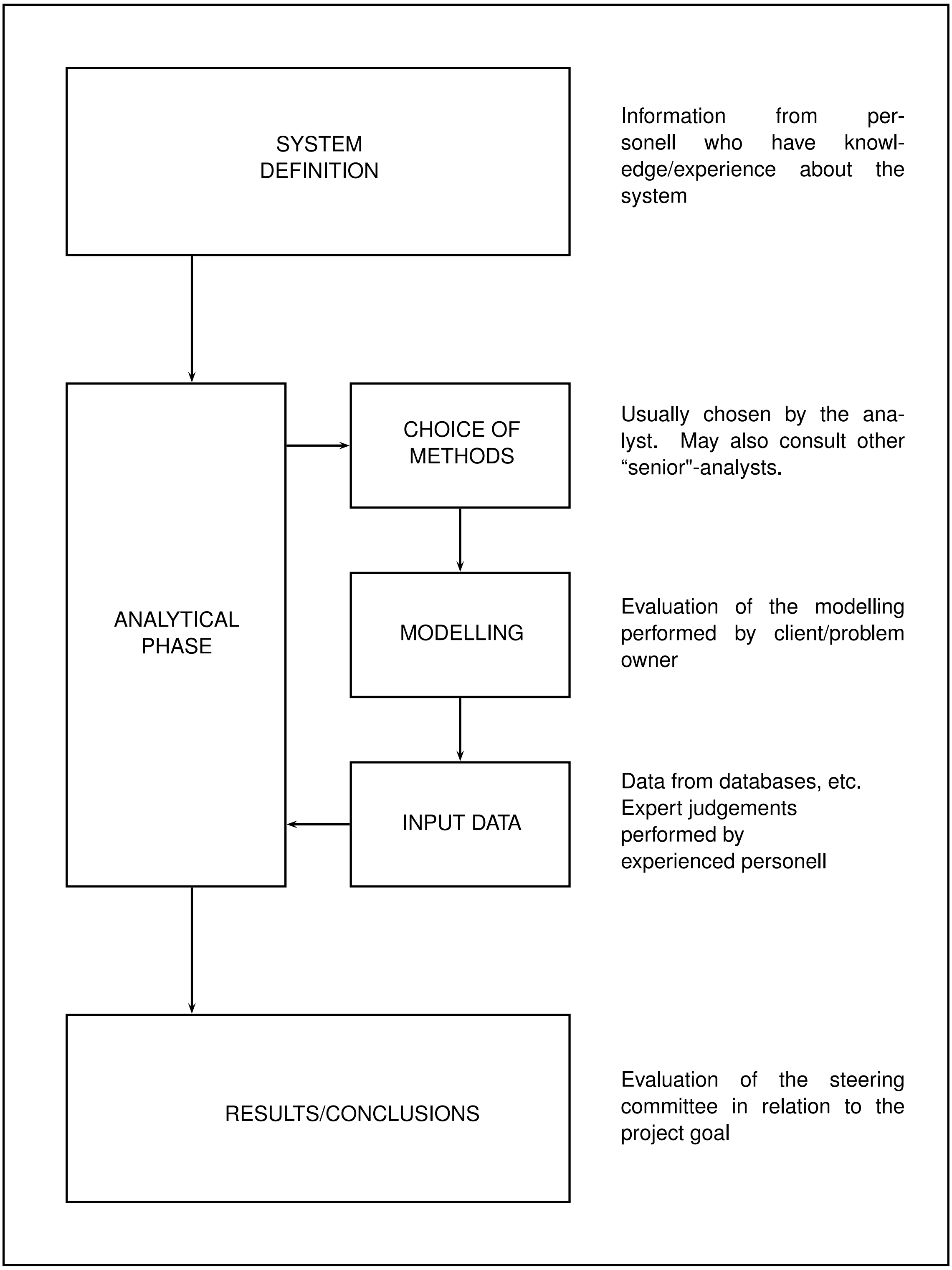


Figure 8.2: Expert judgments of input data compared to other type of information

Figure 8.2 illustrates some of the information that is needed in a project (for instance, a reliability analysis), to show what we mean by "expert judgments" compared to other types of information.

In the system definition phase, we have a need for *information about the system*, how it works, is supposed to be used, maintained, etc. This is information that we may obtain from the personnel who have this experience, and may be viewed as "experts", but this type of information (no matter how important it is) is not what we usually mean by expert judgments.

In the analytical phase, the analysts have to choose *the methods* that will be used, based on the type of problem they are facing. Here, the analysts may be viewed as "experts" of the method, or may consult senior personnel who are perceived as experts of analytic methods, so that these will choose the best methods to the relevant problem. This is also important information, however, neither this is what we usually mean by expert judgments.

When the method is chosen, the problem must be adjusted to the methods. That is, *modelling the problem* by making simplifications and assumptions. Such a modelling (conversion) of the problem should be evaluated by a responsible person within the clients organisation (the "problem owner") who has basic understanding of the methods that will be used, and who may evaluate whether the simplifications and assumptions are valid and acceptable. Such a verification of the modelling is of great importance (and may easily be disregarded or handed over to the analysts only). This type of information/evaluation is neither what we usually view as an expert judgment.

Input data to our models may either be statistical data (objective /"hard" data) or subjective ("soft") data provided by personnel who have the necessary knowledge and experience to provide this information. *Such subjective data estimates is what we here view as expert judgments.* Also, we have restricted this to quantitative data. However, in general also qualitative estimations, as for instance ranking, may be viewed as expert judgments.

Final *results and conclusions are evaluated* by a steering committee/client against the project targets. Such an "expert judgment" of the results is also important, but is not what we usually mean by the term expert judgments.

Evaluations/*quality assurance* performed by the person(-s) responsible of the quality assurance are not illustrated in the figure, but should be performed in all phases of the project. Neither such an evaluation is what we usually mean by expert judgments.

To summarise, we may say that the type of evaluations that are viewed as expert judgments in this case is:

Quantitative estimates of input data to models

8.2.3 Why is there a need for expert judgments?

Expert judgments are primarily of interest when data is lacking, or when we have insufficient statistical data. They are not meant as an alternative to statistical data. Ideally, we should preferably establish databases to obtain statistical data. However, in anticipation of databases being established and events to occur, we have to use the knowledge and experience that are in the possession of persons who work with the system. Otherwise, the problem is not solved. Besides, for new systems and for very rare events, a sufficient base of statistical data will not exist.

It may be claimed that experts have always been utilised in order to give estimates for quantitative values of different kinds, because statistical data have not existed, or have not been easily available. However, this has been done in an informal way. It could be questioned whether this is actually satisfactory. Why formalise this too much ? Why do we not just continue using "engineering judgments" when quantitative values in our models/calculations are needed?

This is not necessarily a black and white situation. There are many reasons not to perform an expert judgment in a too formalistic and extensive way. On the other hand, there are good reasons to be sceptical to an informal handling, such as using the so called "engineering judgments". Advantages and disadvantages of both "extremes" are discussed in Table 8.1.

Table 8.1: Advantages and disadvantages of strictly formalised expert judgments vs. informal engineering judgments.

Factors	**Formal expert judgment**	**Engineering judgment**
Structure	Systematic and structured method/process.	Unsystematic and unstructured process. "Discussion across the table".
Specification of information	Well specified. Only information given as answers of well defined questions.	Imprecise. Assumptions are not specified.
Documentation	All steps of the procedure are well documented.	Poor or none documentation.
Extent of collected information	Limited. Only that obtained through predefined questions.	Wide. May cover many aspects of the subject, also through follow-up questions.
Evaluation of experts	"Objective" rules for evaluation and possible weighting of the experts.	The confidence in a specific expert is judged subjectively by the analyst.
Simplicity	Extensive and expensive.	Very simple. Performed without preparations.

Some of the differences are discussed in the following.

Documentation

By using a formal expert judgment, all the steps in the process are well documented. Normally, by using engineering judgments, very little is documented. A more structured method, as expert judgments, also promotes documentation of the results. Further, the information that is collected often is less precisely defined by engineering judgments. Neither, important assumptions that make a basis for the judgments are not necessarily stated.

Extent of information (Completeness)

The extent of the information collected via formal expert judgments are often limited by only being a response to well defined questions. It does not intercept any "extra" information than the answers to the questions, thus making demands to the questionnaire being worked out. There is little room for follow-up questions and improvisation, something that may impair the completeness of the information gathered about the subject. In engineering judgments, one may more easily gain a general insight in addition to precise quantification of a given subject.

Objective evaluation of the experts

In formal expert judgments, unequal weighting is not allowed if there are no rational reasons for this (as for example use of control questions). Therefore, this method promote a objective handling of the experts, while for engineering judgments, it is often a matter of subjective opinion in who the analyst chooses to have the greatest confidence in.

Simplicity

With its lack of structure and preparation, engineering judgments are often too simple to become credible. Formal expert judgments have a tendency to become quite complex and extensive. Thus, it is a problem that too extensive expert judgments hardly make a relevant alternative to engineering judgments, due to high costs.

Thus, a conclusion is that nowadays' engineering judgments have a big problem in respect to documentation and objectivity. As for how complete the information gets, the picture is more mixed. The advantage of having a well structured list of specified questions is acknowledged. However, this can lead to a non-flexible elicitation, and is not necessarily the best in all situations.

8.2.4 Who are the experts?

The experts are persons with knowledge and experience about the system we want information about. This could for instance be maintenance personnel, so one does *not* necessarily have to have higher education (e.g. University Degree) to be viewed as an "expert".

In the literature this question of "who is the expert" is considered to a very limited extent. Svenson [28] discusses this based on Shanteu, where strong criteria are set to which experience and qualifications a person must have in order to be viewed as an expert. These are:

1. Experience in performing judgments and making decisions
2. More than 10 years of experience within the current subject

3. Inherent qualities like self-confidence and adaptability

A person that satisfies the first or the two first criteria, but not the last, is evaluated by Shanteau to be a novice and not an expert. In his evaluation, such a person will not be able to train to satisfy item iii). Thus, he will stay a novice, even though he sees himself as an expert, and also if the people around him do so.

To us, this is a too rigid judgment of what is required to characterise a person as an expert. We do not ask for "experts" meeting some fixed criteria, but rather the persons having as much knowledge/experience on the subject as possible. Thus, we have quite a pragmatic view to what we mean by the notion "expert".

8.2.5 How should the expert judgment be carried out?

The method of this chapter is based on some basic requirements some of which were also discussed in Section 8.2.3, were we performed an evaluation of expert judgments vs. engineering judgments. The basic requirements are:

1. Documentation
2. Objectivity
3. Empirical control (The estimates should be able to be empirically controlled)
4. Completeness
5. Simplicity

The requirements 1-4 are necessary to obtain scientific credibility. Requirement number five about simplicity is included to secure a practical method that leads to a widespread use.

8.2.6 Terms and notions

In this section we define some terms and notions related to expert judgments. Many of these describe qualities or attributes of the experts or the estimates they provide.

In the presentation below we will always assume that there exist some "true" values for the quantities of interest. In order to make this meaningful we then have to distinguish the random nature of the problem at hand, and the underlying (true) construct of the problem. Let X be the quantity in the real problem which is of interest, e.g., the duration of a critical activity. Due to a set of factors it is not possible to state the actual value of X, and we therefore treat X as a random quantity (stochastic variable) to represent the variability of the problem. In many cases it might make sense to consider the problem as a repetitive problem where X could be seen as realisation of an underlying probability distribution. The parameters in this underlying distribution are, however, unknown. The objective of expert judgements is then to reveal by an elicit process these parameters. These underlying parameters are considered to exist, in the sense that we talk about true underlying parameters. For example, if X is PERT distributed, the quantities L, M and H are considered to be such true underlying parameters of intereset.

Unbiasedness

Unbiasedness may be defined as the degree of "accuracy" of the assessments, and describes to what extent the assessments show a *systematic* deviation from the true value. Systematic deviations may be caused by psychological factors, or by that the experts' experience somehow is not representative for the system in question (e.g. having different operational conditions than the expert is familiar with).

A definition of biasedness may be:

Biasedness = The degree of systematic deviation from the true value

A common *measure* of the experts' biasedness is:

Bias = mean of the estimated values - true value

Calibration

Calibration may have two meanings. Calibrate (as a verb) means to correct the estimates of an expert that shows systematic deviations (giving several estimates), that is provide biased estimates. This corrected (calibrated) estimate will then be the input to the analysis.

Calibration (as an adjective) is used by e.g., Cooke [18] to describe the accuracy of one specific estimate (independent of systematic or random errors). Cooke defines calibration as:

Calibration = to what extent the estimated probability agree with the observed relative frequency

Thus, calibration (as an adjective) is a characteristic of one estimate (the distance between the expert's best estimate and the true value).Unlike unbiasedness (which is applied having more than one estimate), poor calibration is not necessarily due to biasedness (it might be unsystematic). In this memo we use the term 'calibration' as a verb.

Over- and underestimation

In relation to biasedness, notions like over- and underestimation are often used. Overestimation means that the experts in too many cases give an estimate that is higher than the true value. Underestimation means that the expert too often gives an estimate that is too low compared to the true value. If this is done systematically, there is a bias. However, if we calibrate the experts' estimates, this will give very important information. The more systematic the deviations are, the better will the calibrated value become.

Informativeness

Even though the experts do not show any systematic deviations from the correct values, this does not mean that he always gives the correct values. He may even have random/unsystematic deviations, which result in spread or variation of the estimates round the true values. Here we use the notion informativeness to describe the experts' unsystematic (unpredictable/random) deviation from the correct values. This is also referred to as the *degree of precision*.

Informativeness = the degree of unsystematic variation/spread of the estimates around the true values. (High informativeness = low variation).

An expert that has great variations in his estimates will, even though he sometimes strikes the target, be perceived to have little informativeness (is not to "be trusted"), and should be given small weight when his estimates are weighted with the other experts' estimates.

One way to measure the informativeness is to use the sum of squares of the deviations between the experts' estimates and the true values.

NOTE! Some authors (e.g., [18]) use informativeness about the experts' own evaluation of how certain he is of his estimate, that is, the confidence interval he puts around his best estimate. This we will denote "subjective informativeness", see below.

Subjective informativeness

Subjective informativeness = the informativeness (degree of precision) of the estimate, as assessed by the expert himself

This subjective informativeness is given by the confidence interval he puts around his best estimate. For example in case of PERT distributed random quantities in the model, we ask first the expert to assess the values of L, M, and H. Then for each of these, we ask the expert to assess the informativeness regarding these values. For example for the parameter$M$,the expert gives M_{EV} (expected value for M), M_{LV} (low value for M), and M_{HV} (high value for M). $M_{\mathrm{HV}} - M_{\mathrm{LV}}$ is an expression of how confident the expert is in his own estimate.

Note that it has been shown not to be a good correlation between an expert's calibration ("accuracy") and his subjective informativeness (self-confidence). Thus, there is no guarantee whatsoever that an expert who gives narrow confidence intervals gives better estimates (more accurate) than one with wide confidence intervals.

Over- and underconfidence

Over- and underconfidence are characteristics of the expert's confidence in his own judgments.

Overconfidence means that the expert has a too great confidence in his own estimates, by giving too narrow confidence intervals around his best estimate. Underconfidence means that the expert gives too wide confidence intervals around his best estimate.

Dependence

Dependence is often related to biasedness. If the expert gives biased estimates and this biasedness shows special patterns, we talk about dependencies. For instance, the biasedness may increase when the correct values increase. This represents a positive dependence (correlation) between the correct values and the bias. (Dependence may be seen as a special type of biasedness.)

Resolution

Resolution = the ability to separate the probability of a specific event from the average probability for the total set of events

An example of this is the ability to estimate the most likely duration of the given project with it's characteristics compared to a general project.

Consistency

Consistency = the assessment is independent of the method/approach that is used and also of when the evaluation is carried out

Consistency means that the judgments are reproducible, that is, we get the same result independent of time and method. For each expert this means that he is consistent if e.g. he provides the same result/estimate, even though the question is phrased in different ways, and that he gives the same answer if the assessment is repeated at a later stage of the elicitation. If the expert is inconsistent, he should be given a lower weight than the other experts, when the overall estimate is calculated.

Coherence

Coherence = conformity with the laws of the theory of probability

This may also be referred to as "logical consistency".

For example, according to probability theory, Pr(A or B) = Pr(A) + Pr(B) - Pr(A and B), and Pr(A^*)= 1 - Pr(A). If the expert does not provide answers that conform with these rules, he is said to be incoherent. (If the experts are not coherent, the analyst should avoid using questions that include probabilities.)

Reproducibility or Inter-expert reproducibility

Reproducibility = the extent to which different experts provide similar results to a given question (when using the same method)

Lack of reproducibility should indicate that at least some of the experts do not give satisfying estimates.

Reproducibility may be measured through the spread.

8.3 Check list for expert judgment exercises

In the following we present a checklist for expert judgement exercises where the most important recommendations/warnings are given.

This may be used as a check-list for relatively simple expert judgments, where individual weighting of the experts is not to be used. It is not recommended to carry out calibration and weighting of the experts (in phase 3) during simple expert judgments.

To get a further explanation of each step in the procedure, and the use of calibration and weighting of experts, we refer to the next chapters.

8.3.1 Checklist for phase I: The preparation phase

1. Describe the problem, what is to be estimated
2. Avoid too narrow delimitation of the problem
3. State the reason for the need for experts judgments in relation to the existing problem
4. Estimate the available budget for executing the expert judgment
5. Appoint a time for when the expert judgment must be executed
6. Indicate how much time each expert must set aside
7. Determine the number of experts to be used
8. Possible statistical data should be exhibited to the experts
9. Choose the experts with most knowledge/experience about the problem of interest
10. Choose the experts who are available and motivated
11. Avoid having too many experts with a great interest in the project (interest of a specific result)
12. Avoid particular dominating persons of high status
13. The client should control the list of experts
14. Gather information about the experts' background and experience
15. Decide whether individual estimating or a group process is to be used
16. Formulate an estimation method/ a group technique, respectively
17. Give an attractive format for the questionnaire
18. Formulate clear and simple questions
19. Explain to the experts the difference between uncertainty (variability) in the quantity to assess (e.g., activity duration), and the uncertainty in the parameter estimates
20. The questions must be logically correct and understandable
21. Use graphics to indicate uncertainty (high/low estimates)
22. Decide on suitable level of decomposition of the problem
23. Carry out a test of the questionnaire on colleagues

8.3.2 Checklist for phase II: The elicitation phase

1. Explain to the experts the problem and what is to be estimated
2. Explain to the experts how the expert judgments is to be executed
3. Explain how the results will be treated
4. Go through the forms the experts shall fill in
5. Emphasise the importance of the experts being sincere and honest when giving the estimates
6. Call on the experts to not be affected/dominated by the other experts or by the analyst/process leader
7. Explain to the experts some of the most important phenomenon which causes biased/incorrect estimates
8. The experts must be treated in a fair and correct way
9. The analyst/process leader must remain neutral and not actively take part in the evaluations so that the estimates are affected
10. The analyst/process leader must be present during the elicitation to show interest, control, and to answer questions
11. Estimate the duration of the questioning. (By individual estimation, max. 1 hour before having a break).
12. The analyst/process leader must see to that not one (or a few) expert dominates or affects the others
13. If only one expert is used, the estimates must be well-founded
14. For each parameter of interest, be as specific as possible to describe the cost element, the activity etc for which parameters are required
15. Ask the experts to first indicate their lowest and highest estimate. Thereafter, their best estimate
16. If the variable of interest is PERT distributed, it is recommended to ask the experts to estimate the most likely value (M), the P10 and the P90 percentiles[1].
17. To assess the subjective informativeness ask the expert to give low and high values for each of the three estimates in a typical case (base case)
18. For subsequent quantities subjective informativeness is assessed by L, E and H where L means less subjective informativeness (more uncertain than in the base case), E means equal subjective informativeness, and H means higher subjective informativeness (more certain than in the base case).

[1]The Px percentile is defined by $\Pr(X \leqslant Px) = x\%$. For example $\Pr(X \leqslant P10) = 10\%$

19. Equal weights of the experts should be used when there is no strong argument for weighting
20. Weighting is not recommended for simple expert judgments
21. Finally, ask the experts if there are any questions that are not clear, so that these may be clarified before ending the elicitation
22. In a group processes (after ind. proc.) the experts should affirm for that the result is complete, before finishing the process.

8.3.3 Checklist for phase III: The calculation phase

1. Calibration of the experts is not recommended for simple expert judgments
2. Unequal weighting is not recommended for simple expert judgments
3. Establish a common estimate based on the experts' estimates
4. The analyst must document all steps in the process
5. All assumptions (made by both analyst and experts) should be gathered in a separate form
6. The reason why each expert is chosen, plus his strengths and knowledge of the current subject, must be documented.
7. The "raw-estimates" made by the expert must remain available.
8. Any form for manipulation/weighting of the estimates must not be carried out without including argumentation/documentation
9. All calculations must be reproducible
10. The experts' names should be stated, but each single estimate does not need to be allocated to a specific expert, if anonymity is considered to encourage objectivity
11. Present the results to the client; how results are achieved, and who the experts are.
12. Graphical presentation of the results may be advantageous, especially to show the uncertainty.
13. The results, including the handling of each single estimate are presented to the experts (may be in writing)
14. Evaluate whether the results seem reasonable and accurate.
15. Evaluate whether there is need for further work (more data)
16. File the case/project with all documentation.

17. Update possible databases/bases of knowledge

18. Make a "debriefing" of the expert judgment process (how did it work?)

19. Correct the items in the procedure for carrying out the expert judgment, when weaknesses are experienced.

8.4 Calculation aspects

8.4.1 Criterion for performing calibration

Calibration is carried out if there are distinct indications of the expert systematically over/under-estimating the correct value, such that valuable information is lost if he is not calibrated. (Unless he is calibrated, he will be assigned a very low weight, representing a loss of information.). Calibration requires the use of control questions (seed variables). Generally there should be strong evidence for calibration, which requires several control questions. Now, introduce:
n = The number of control questions
x_i = the correct value (control question no. i), known by analys
Y_i = the expert's estimate
Z = The number of $Y_i - x_i$ (control question no. i) that are >0

We will calibrate the expert when Z is either near to 0 or near to n. The following simple rules are suggested:

(1) For $n \geqslant 5$, a calibration is done when

$Z < n/2 - \sqrt{n}$ or $Z > n/2 + \sqrt{n}$

Example: For $n = 5$ we calibrate if Z = 0, or Z = 5

For $n = 10$ we calibrate if Z = 0, 1, 9, 10

(2) For $2 \leqslant n \leqslant 4$ and $Z = 0$ or $Z = n$, a calibration is done if, in addition, Y_i/x_i generally are "large" (»1) or "small" («1). That is:

(i) Z = 0 and $1/n \sum_i (Y_i/x_i) < 1/(6\text{-}n)$, or

(ii) Z = n and $1/n \sum_i (Y_i/x_i) > (6\text{-}n)$

Example 8.1 Example cost estimate
Assume $n = 3$ control questions x_i, ($i = 1, \ldots, 3$) for cost estimation of the activities A, B and C. Assume the estimates and correct values shown in Table 8.2:

Table 8.2: Estimates and correct values from 3 activities, one expert

Activity	x_i (true, known value)	Y_i (estimate)
A	50	70
B	60	90
C	70	140

All Y_i in this case are larger than x_i, that is, $Z = 3$, but: $1/n \sum_i (Y_i/x_i) = 1/3$ (7/5 + 9/6 + 14/7) = 1.63 <(6 - n) = 6 - 3 = 3. That is, here it is chosen not to calibrate. □

8.4.2 Method for performing calibration

In the following we describe the principle for calibration if the criterion for calibration is satisfied. We assume that there is a linear relation between the true values (x_i's) and the estimates (Y_i's):

$$x_i = \beta_0 + \beta_1 Y_i + \text{error term} \tag{8.1}$$

Note that here we write x as a function of Y and not the other way as we usually do. To estimate β_0 and β_1 we apply standard LS methods, i.e.:

$$\hat{\beta}_1 = \frac{\sum_i (Y_i - \bar{Y})x_i}{\sum_i \left(Y_i - \bar{Y}\right)^2} \tag{8.2}$$

$$\hat{\beta}_0 = \bar{x} - \hat{\beta}_1 \bar{Y} \tag{8.3}$$

For a given value of Y, say y, we now estimate the parameter of interest by:

$$\hat{x} = \hat{\beta}_0 + \hat{\beta}_1 y \tag{8.4}$$

Note that if Y is less than $Y_{i,\min} = \min_i Y_i$ it is recommended to use a regression line through the origin, i.e.,

$$\hat{x} = \left(\frac{\hat{\beta}_0}{x_{i,\min}} + \hat{\beta}_1\right) y \tag{8.5}$$

Example 8.2 With the data given above we find (although calibration was not recommended):

$$\hat{\beta}_0 = 33.077 \text{ and } \hat{\beta}_1 = 0.27$$

Assume now that we ask the expert to estimate a new value for duration of a new activity D (M = Most likely value). He assigns the value $y = 100$. The estimate is now given by:

$$\hat{x} = \hat{\beta}_0 + \hat{\beta}_1 y = 33.077 + 0.27 \cdot 100 \approx 60 \tag{8.6}$$

□

8.4.3 Weighting of experts

Various principles for weighting of experts exist. In the following we describe some of these.

Equal weighting

Give the experts a weight $1/m$, where m = the number of experts. Equal weighting of the experts is used when there is no basis for performing differentiation of weighting (e.g. control questions, mutual weighting, knowledge profile.)

Based on control questions

Give the experts a weight related to the estimated random error (variance) obtained from evaluation of the control questions. This expresses the experts' "true informativeness". Note that we now write the relation between x and Y in the normal way:

$$Y_i = \alpha_0 + \alpha_1 x_i + \text{error term} \tag{8.7}$$

where the objective is to estimate the variance of the error term. To find the variance of the error term we estimate the regression line, let SS be the square sum of the residuals for expert k:

$$SS_k = \sum_i (y_i - \hat{\alpha_0} - \hat{\alpha}_1 x_i)^2 \tag{8.8}$$

i.e., the differences between the observation (y_i's) and the estimated line. An estimate for the variance is given by:

$$S_k^2 = SS_k/(n-2) \tag{8.9}$$

where n is the number of control questions given to expert k. If we use e.g., MS Excel to estimate the parameters we directly find S as the "Standard Error". If we have m experts the weight of expert k is then

$$w_k = \frac{S_k^{-2}}{\sum_j S_j^{-2}} \tag{8.10}$$

where S_k is the estimated standard deviation of expert k.

Based on mutual evaluation

Give the experts a weight related to how they where evaluated by the other experts. All the experts evaluate the other experts by giving them points (p) from 0-10 (10 is maximum).

The weight for each single expert is then the sum of scores which the expert achieves divided by the total score given by all the experts. (Thus, the sum of the weights equals 1.0). The setup is shown in Figure 8.3.

Weight for expert k is given by:

$$w_k = \frac{\sum_j p_{j,k}}{\sum_j \sum_{i \neq j} p_{j,i}} \tag{8.11}$$

Based on knowledge profile

Give the experts a weight based on their knowledge. For example, the number of years as project manager, the experience with similar type of projects etc.

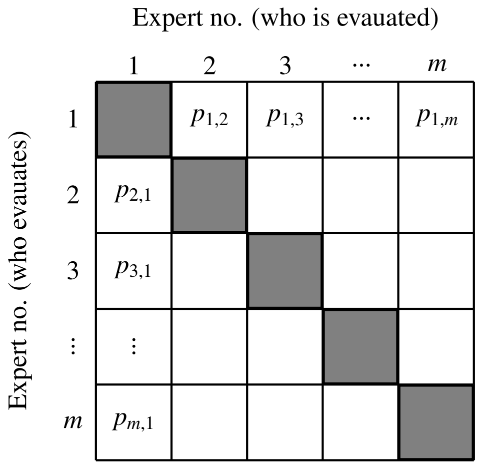


Figure 8.3: Mutual evaluation of experts

Based on arguments/documentation used in the discussion

Give the experts a weight related to their argumentation for their estimates. In the argumentation, personal experiences and references to sources, among other things, may be included. (Such an “expert judgement” of the experts performed by the analyst, must be substantiated and documented if it is to be used.)

8.4.4 Standard weighting model - Experts only

Assume we have m experts that have made individual statement, $\hat{x}_j$ regarding the parameter of interest. Further assume that weights w_i are established for each expert according to procedures described in Section 8.4.3. The combined estimate of the parameter to estimate (x) is now given by:

$$\hat{x} = \sum\nolimits_{j=1:m} w_j \hat{x}_j \tag{8.12}$$

8.4.5 Transforming percentiles

It is recommended to ask the expert to assign percentiles in the PERT distribution rather than the extreme values L and H.Assume that the weighting model in equation (8.12) is applied for all parameters, and the results are denoted $\hat{P}_{10}$, $\hat{M}$ and $\hat{P}_{90}$. We now seek $\hat{L}$and $\hat{H}$. If we have implemented the CDFPert() function we should then require:

$$\text{CDFPert}(\hat{P}_{10}, \hat{L}, \hat{M}, \hat{H}) = 10\% \tag{8.13}$$

and

$$\text{CDFPert}(\hat{P}_{90}, \hat{L}, \hat{M}, \hat{H}) = 90\% \tag{8.14}$$

And try to solve these two equations wrt $\hat{L}$ and $\hat{H}$.

Example 8.3 Assume that $\hat{P}_{10} = 6$, $\hat{M} = 10$ and $\hat{P}_{90} = 14$. By applying the pRisk.xls program we may now generate one cell containing CDFPert($\hat{P}_{10}$, $\hat{L}$, $\hat{M}$, $\hat{H}$), and one cell containing CDFPert($\hat{P}_{90}$, $\hat{L}$, $\hat{M}$, $\hat{H}$). Then we set a third cell equal to the sum of these two cells. By using the solver requiring the sum to be one, and for example the cell with CDFPert($\hat{P}_{10}$, $\hat{L}$, $\hat{M}$, $\hat{H}$) to be equal to 0.1 in the "Constraints" editor, we find $\hat{L} = 2.11$ and $\hat{H} = 17.89$. Observe that the CDFPert($\hat{P}_{90}$, $\hat{L}$, $\hat{M}$, $\hat{H}$) cell evaluates to 90%. □

8.4.6 Standard weighting model - Experts and data

In some situations we both have data and experts judgements. In principle we may use Bayesian updating strategies to find the final estimates to apply. In the following we propose a slightly simpler approach where we weight the expert judgement results with the estimates found by statistical analysis of data.

Weight of experts

Equation (8.12) is used to find the combined estimate of the parameter of interest. One way to weight this value with the data is to calculate the "variance" of the estimate in equation (8.12). An unbiased estimator for the variance based on the variation of the expert statements is:

$$S_{VE}^2 = \frac{1}{1 - \sum_{j=1}^{m}} w_j^2 \sum_{j=1}^{m} w_j \left(\hat{x}_j - \hat{x}\right)^2 \tag{8.15}$$

If there are few experts this measure is not very reliable. A better approach may then to use the "Subjective informativeness" assessed by the experts. Now, assume that each expert, k, has stated an estimate, $\hat{x}_k$, but also low ($\hat{x}_{k,L}$) and high ($\hat{x}_{k,H}$) values for the estimate. Assume that the low and high values corresponds to the P_{10} and P_{90} percentiles. If we assume that the underlying uncertainty distribution of the expert is PERT distributed, we may apply the approach shown in the prvious example to find the L, M and H values of the uncertainty distribution, and hence the variance by standard formulas. A more pragmatic approach would be to claim that the standard deviation is proportional to the distance between the low and high value, and then try to assess the proportional constant. Following the example we then assess the "self evaluated standard deviation" by :

$$\hat{S}_k = 0.37(\hat{x}_{k,H} - \hat{x}_{k,L}) \tag{8.16}$$

If $\hat{S}_k^{-2}$ is used as basis for the weighting of expert k, and the $\hat{S}_k^{-2}$'s are considered as true variances, it can be shown that the variance of the weighted means equals the reciprocal of the sum of these variances, and hence a reasonable estimate for the variance of the weighted estimate based on all experts is found by:

$$S^2_{\mathrm{SE}} = \frac{1}{\sum_{j=1:m} \hat{S}_j^{-2}} \tag{8.17}$$

If there are few experts it is recommended to apply equation (8.17). If there are a medium number of experts, say 3 to 5 one may calculate the variance by both equation (8.15) and equation (8.17) and use the maximum of these two values. Generally we denote the estimate of the variance of the expert judgement estimator by S^2_{E}.

Weight of data

In some cases it is easy to find the variance of the estimator used when statistical data is available. For example the variance of the estimator for the mean value in the normal distribution is found by the sample standard deviation divided by the square root of the number of observations. In more complicated situations we may use the principle of *bootstrapping* to find the variance of the estimator. The procedure is as follows: Let $z_1, z_2, \ldots, z_n$ be observations from the distribution for which we are seeking estimates. Apply an estimation procedure to find the parameter vector. Denote the result by $\hat{\theta}$. Now repeat, $k = 1, 2, \ldots$

1. Generate n pseudo random numbers from the actual probability distribution with parameter vector $\hat{\theta}$
2. Apply the estimator again to find a new estimator for the pseudo generated numbers, and let the result be denoted $\hat{\boldsymbol{\theta}}_k$ for the k^{th} iteration

The sample variance of the j^{th} element of the $\hat{\boldsymbol{\theta}}_k$-sequence may now be used as an estimate for the variance of the estimator based on data. Let S^2_D denote the variance based on the data.

Combining it

If we let $\hat{x}_E$ and $\hat{x}_D$ denote the estimates from combined experts and data respectively, and further S^2_E and S^2_D the corresponding estimated for variances, we find the final weighted estimate of experts and data by:

$$\hat{x} = \frac{S_E^{-2}\hat{x}_E + S_D^{-2}\hat{x}_D}{S_E^{-2} + S_D^{-2}} \tag{8.18}$$

8.5 Worked example

Assume that we have three experts. They are all calibrated with three control questions shown in Table 8.3:

All experts are asked to give the most likely value (M) for a new activity D, with corresponding uncertainties in the estimate shown in Table 8.4:

Table 8.3: True values and estimates from three experts

Activity (Control)	x_i(true, known value)	Y_i (Expert 1)	Y_i (Expert 2)	Y_i (Expert 3)
A	50	70	40	60
B	60	90	80	50
C	70	140	60	90

Table 8.4: New values and self assessed uncertainty by the experts

Activity	Y(M in PERT)	Low $Y(\mathrm{P}_{10})$	High $Y(\mathrm{P}_{90})$
Expert 1	100	75	150
Expert 2	80	60	95
Expert 3	60	55	70

Observation from previous project shows the following costs: 53.2, 60.4, 52.4, 66.8, 56.2, 72.2, 71.7 and 73.2.

Find a combined estimate from the experts, and an estimate from the data. Then combine these two estimates into a final estimate.

We decide to calibrate the expert independent of the “calibration test”. Regression analysis from MS Excel is shown in Table8.5.

Parameter estimate	Expert 1	Expert 2	Expert 3
$\hat{\beta}_0 = \bar{x} - \hat{\beta}_1 \bar{Y}$	33.08	45.00	36.92
$\hat{\beta}_1 = \frac{\sum_i (Y_i - \bar{Y}) x_i}{\sum_i \left(Y_i - \bar{Y}\right)^2}$	0.27	0.25	0.35

Predictions of a new value for a new activity, D, by the experts are shown in Table 8.5. Low and high values, specified by the expert is given in Table 8.6.

Weighted calibrated prediction based on self evaluated weights:

$$\hat{x} = \sum_{j=1} m w_j \hat{x}_j = 59.61$$

Corresponding variance of this weighted average:

$$S^2_{\mathrm{SE}} = \frac{1}{\sum_{j=1:m} \hat{S}_j^{-2}} = 2.60$$

Table 8.5: New values predicted by the experts with calibrated values

Quantity	Expert 1	Expert 2	Expert 3
Y = Prediction by expert	100	80	60
$\hat{x} = \hat{\beta}_0 + \hat{\beta}_1 y$	60.00	65.00	57.69

Table 8.6: Low and high values specified by the expert and derived results

Quantity	Expert 1	Expert 2	Expert 3
Low Y by expert =P_{10}	75	60	55
High Y by expert =P_{90}	150	95	70
Calibrated low, ($P_{10} = \hat{x}_{k,L}$)	53.27	60.00	55.96
Calibrated high, ($P_{90} = \hat{x}_{k,H}$)	73.46	68.75	61.15
$\hat{S}_k = 0.37(\hat{x}_{k,H} - \hat{x}_{k,L})$	7.471	3.238	1.921
$\hat{S}_k^{-2}$	0.018	0.095	0.271
$w_k = \frac{\hat{S}_k^{-2}}{\sum_{j=1}^{m} \hat{S}_j^{-2}}$	0.047	0.248	0.705

Chapter 9

Optimization of turnaround activities

9.1 Introduction

Turnarounds are scheduled events wherein an entire process unit of an industrial plant such as an oil & gas production platform is taken shut-down for an extended period for modification and/or renewal. Sometimes the term 'revision stop' is used to denote a turnaround.

Turnarounds are expensive - both in terms of lost production while the process unit is offline and in terms of direct costs for the labour, tools, heavy equipment and materials used to execute the project. Turnarounds have unique project management challenges due to the large number of activities to execute, logistics for labour and spare parts, and safety issues due to many activities in parallel.

In this chapter we mainly discuss how maintenance activities either could be conducted individually, or if they should be in cooperated in a turnaround to save cost since the production already is down.

9.2 Single component considerations

In this section we consider the situation where only one maintenance activity is considered. The question to be answered is whether a maintenance activity should be carried out at its "optimal execution time", or if it should be carried out as a part of a turnaround (revision stop).

9.2.1 Preventive maintenance

Due to wear and tear of components their failure probability often increases with component age. In order to reduce the likelihood of failure components are often preventively maintained. For example a switch machine in a railway system is in Norway tentatively overhauled every six year. Such an overhaul activity includes replacement of components exposed to wear, lubrication, cleaning etc. In maintenance theory it is made a distinction between age or calendar based activities, and condition based activities. Historically, preventive maintenance was carried out based on the age of a component. The only trigger for maintenance was age, running hour, millage run for a car etc. Since the correlation between age and failure is rather vague, one seeks to find more precise indicators that could be correlated to the actual failures of the component. This has led to increased used of condition monitoring techniques where the objective is to utilize the condition of a component as an indicator whether a failure will occur in the near future. An example of a condition based activity is replacement of a rail when cracks of critical length are found by ultrasonic inspection. The main rationale for preventive maintenance is that it is cheaper to prevent a failure from occurring by a preventive maintenance task than it will be if a failure occurs. A range of optimization models have been derived in order to determine the appropriate level of preventive maintenance. The reason why maintenance is addressed within project risk management is that turnarounds essential is a huge package of preventive maintenance activities organized as a project.

9.2.2 Single activity - Preventive maintenance not included in the turnaround

Consider a component where a preventive maintenance (PM) action is conducted at predetermined intervals due to an increasing hazard rate[1], $z(t)$. In order to find an optimal interval for a maintenance action we may establish the average cost per time unit as a function of the maintenance interval, say τ:

$$C(\tau) = C_{\mathrm{PM}}/\tau + \lambda_{\mathrm{E}}(\tau)\,[C_{\mathrm{CM}} + C_{\mathrm{EP}} + C_{\mathrm{ES}}] \tag{9.1}$$

where C_{PM} is the cost of a preventive maintenance action (to prevent failures), C_{CM} is the cost of a corrective maintenance (CM) action (given that a failure did occur), $\lambda_{\mathrm{E}}(\tau)$ is the effective failure rate, i.e., the expected number of failures per time unit when the component is preventively maintained every τ time unit, C_{EP} is the expected production losses upon a component failure, and finally C_{ES} is the expected safety cost upon a component failure, including material damages and environmental losses.

In the following we let $C_U = C_{\mathrm{CM}} + C_{\mathrm{EP}} + C_{\mathrm{ES}}$ denote the expected unplanned cost upon a failure. The effective failure rate depends on the life time distribution of the component. The Weibull distribution is a widely used distribution for aging components. In the case of Weibull distributed life times we may find approximation formulas for the effective failure rate. If we know the mean time to failure , MTTF (without maintenance), and the aging parameter (α) of the lifetime distribution of the

[1]The hazard rate is given by $h_X(x) = \frac{f_X(x)}{1-F_X(x)}$, where $f_T(t)$ is the probability density function of the lifetime T of the unit, and $F_T(t)$ is the cumulative distribution function. The hazard rate is also to be understood as the conditional probability of a failure in $(t, t + \Delta t)$ given that the unit has survived up to time t. In some textbooks the hazard rate is also dented the failure rate function.

component, the effective failure rate may be approximated by:

$$\lambda_E(\tau) = \left(\frac{\Gamma(1 + 1/\alpha)}{\text{MTTF}}\right)^{\alpha} \tau^{\alpha-1} \tag{9.2}$$

where $\Gamma(\cdot)$ is the gamma function. The approximation is good when the maintenance interval is small compared to the MTTF. If the maintenance interval is approaching the MTTF value, the approximation in equation (9.2) is not very accurate, and we might use the following improved approximation:

$$\lambda_E(\tau) = \left(\frac{\Gamma(1 + 1/\alpha)}{\text{MTTF}}\right)^{\alpha} \tau^{\alpha-1} \left[1 - 0.1\alpha(\tau/\text{MTTF})^2 + (0.09\alpha - 0.2)\tau/\text{MTTF}\right] \tag{9.3}$$

In MS Excel the gamma function could be found by EXP(GAMMALN(x)). In the following we will always assume that the approximation in equation (9.2) is sufficient for our purpose. By setting the derivative of $C(\tau)$ in equation (9.1) equal to zero, we find the optimal interval to be:

$$\tau^* = \frac{\text{MTTF}}{\Gamma(1 + 1/\alpha)} \left(\frac{C_{PM}}{C_U(\alpha - 1)}\right)^{1/\alpha} \tag{9.4}$$

9.2.3 Single activity - Consideration for inclusion in turnaround - Static consideration

A turnaround is usually conducted every year, every two years, or every three years. There is an aim to conduct the turnaround in a period where working conditions are good (summer period), and where also production losses are as low as possible which historically also has been in the summer period since energy prices usually reduces when the demand goes down. It is therefore a common practice to conduct the turnarounds in the summer. The interval between turnarounds is denoted τ_{TA}.

The formula for the average cost $C(\tau)$ in equation (9.1) needs to be modified if maintenance is considered. Let C_{TA} be the preventive maintenance cost if the maintenance of the component is included as part of the turnaround. Usually we will have that $C_{TA} < C_{PM}$. We will now investigate whether the activity should be included in the turnaround. The criterion for inclusion in the turnaround is that the average cost is reduced. An example is given to demonstrate the steps. The relevant parameters are given in Table 9.1 where the time unit is months, and cost unit is thousand NOKs.

We first apply equation (9.4) and get $\tau^* \approx 18.3$ months. The total average cost in the situation where the PM activity is not included in the turnaround is thus ≈ 0.455 per month by inserting $\tau = \tau^*$ in equation (9.1). If the PM is included in the turnaround, the average cost is found by inserting $\tau = \tau_{TA}$ and $C_{PM} = C_{TA}$ in equation (9.1). This gives a monthly cost of $\approx 0.43 < 0.455$. It is thus in average cheaper to include the PM into the turnaround. We observe that $\tau_{TA} < \tau^* < 2\tau_{TA}$. We might therefore also consider to insert $\tau = 2\tau_{TA}$ and $C_{PM} = C_{TA}$ in equation (9.1). This gives a monthly cost of $\approx$ 0.44 which is still better than the situation without including the PM in the turnaround, but less favourable comparing by including the PM in every turnaround. The optimal strategy is thus found to be to include the PM activity in every turnaround with the given assumptions.

Table 9.1: Parameters for the decision regarding inclusion in turnaround, $\tau^* > \tau_{TA}$

Parameter	Value
α	2.5
MTTF	48
C_{PM}	5
C_{TA}	4
C_{U}	50
τ_{TA}	12

9.2.4 Single activity – dynamic consideration

In Section 9.2.3 we considered a situation where a static regime was established with respect to finding the appropriate maintenance interval of the component. By "static" we here understand that the same interval applies for the component during the entire "life" of the installation. This is appropriate if we are able to synchronise all maintenance activity from the start-up of the production, and when nothing "changes", in terms of major modifications, increase in reliability etc. In practice, however, things will change all the time, and it is usually required to be able to change strategies "on the fly". This means that we need a dynamic approach. We now consider a situation where we are planning the next turnaround. For the time, we let $t = 0$ represent the time we are going to conduct the next turnaround. Let x be the "age" of the component we are considering. This means that the component was preventively maintained x time units ago. To proceed we continue the example provided in Section 9.2.3, but we now assume that $x = 8$ months. The "due" point of time of PM if we did not include the activity would then be approximately in 18-8 = 10 months. In this case, it is therefore reasonable to postpone the maintenance until the next PM. If we include the PM in the coming turnaround the total cost in the next period (i.e., from now until the PM next year) is:

$$C_{\text{First TA}} = C_{\mathrm{TA}} + \lambda_{\mathrm{E}}(\tau_{\mathrm{TA}}) \cdot C_{\mathrm{U}} \cdot \tau_{\mathrm{TA}} \tag{9.5}$$

If we choose to skip the PM activity for the coming turnaround, and wait till the next we get the following total cost:

$$C_{\text{Second TA}} = \lambda_{\mathrm{E}}(\tau_{\mathrm{TA}} + x) \cdot C_{\mathrm{U}} \cdot (\tau_{\mathrm{TA}} + x) - \lambda_{\mathrm{E}}(x) \cdot C_{\mathrm{U}} \cdot x \tag{9.6}$$

where $\lambda_{\mathrm{E}}(\tau_{\mathrm{TA}} + x) \cdot C_{\mathrm{U}} \cdot (\tau_{\mathrm{TA}} + x)$ is the expected "unplanned" cost from the last PM until the next turnaround, and $\lambda_{\mathrm{E}}(x) \cdot C_{\mathrm{U}} \cdot x$ is the expected "unplanned" cost from the last PM up to now (i.e., what already has been "paid"). With the example data we get:

$$C_{\text{First TA}} = C_{\mathrm{TA}} + \lambda_{\mathrm{E}}(\tau_{\mathrm{TA}}) \cdot C_{\mathrm{U}} \cdot \tau_{\mathrm{TA}} \approx 5.16 \tag{9.7}$$

and

$$C_{\text{Second TA}} = \lambda_{\mathrm{E}}(\tau_{\mathrm{TA}} + x) \cdot C_{\mathrm{U}} \cdot (\tau_{\mathrm{TA}} + x) - \lambda_{\mathrm{E}}(x) \cdot C_{\mathrm{U}} \cdot x \approx 3.73 \tag{9.8}$$

Hence, the PM activity should not be included in the coming turnaround but in the turnaround coming next year. Note that we might consider to execute the PM in between the two coming turnarounds, but the example in Section 9.2.3 indicates that the gain by synchronizing with a turnaround is always the best alternative.

9.2.5 Single activity - "Arctic maintenance"

In some situations it is not possible to conduct a corrective maintenance action in the period between the major shutdowns (turnarounds) of the plant. For example in the arctic it is considered almost impossible to conduct a major repair in wintertime. If a failure occurs in the period between two turnarounds, there will thus be a production loss from that failure until the next turnaround. In the modelling we still assume Weibull distributed lifetimes. For the Weibull distribution[2] we have:

$$f_T(t) = \alpha\beta(\beta t)^{\alpha-1}e^{-(\beta t)^{\alpha}} \tag{9.9}$$

$$R(t) = \Pr(T > t) = 1 - F_T(t) = e^{-(\beta t)^{\alpha}} \tag{9.10}$$

$$\beta = \frac{\Gamma(1/\alpha - 1)}{\text{MTTF}} \tag{9.11}$$

In the modelling we now we still assume that the cost of an unplanned failure is C_{U}. But in addition to the immediate cost of a failure we assume that for the remaining time until the next opportunity for maintenance there will be a cost of C_W per time unit until the component could be repaired.

If we include the PM in the coming turnaround the total cost in the next period (i.e., from now until the PM next year) is:

$$C_{\text{First TA}} = C_{\text{TA}} + [1 - R(\tau_{\text{TA}})]\, C_{\text{U}} + \int_{t=0}^{\tau_{\text{TA}}} f_T(t)\,(\tau_{\text{TA}} - t)\, C_W dt \tag{9.12}$$

If we choose to skip the PM activity for the coming turnaround, and wait till the next we get the following total cost in between the two turnarounds:

$$C_{\text{Second TA}} = [1 - R(\tau_{\text{TA}} + x)/R(x)]\, C_{\text{U}} + \int_{t=0}^{\tau_{\text{TA}}} f_T(t+x)\,(\tau_{\text{TA}} - t)\, C_W dt/R(x) \tag{9.13}$$

We proceed by the same data as in the previous examples but we now also let $C_W = 10$. We then get:

$$C_{\text{First TA}} = C_{\text{TA}} + [1 - R(\tau_{\text{TA}})]\, C_{\text{U}} + \int_{t=0}^{\tau_{\text{TA}}} f_T(t)\,(\tau_{\text{TA}} - t)\, C_W dt \approx 5.93 \tag{9.14}$$

and

$$C_{\text{Second TA}} = [1 - R(\tau_{\text{TA}} + x)/R(x)]\, C_{\text{U}} + \int_{t=0}^{\tau_{\text{TA}}} f_T(t+x)\,(\tau_{\text{TA}} - t)\, C_W dt/R(x) \approx 7.06 \tag{9.15}$$

[2]Note that various parameterization exist for the Weibull distribution, and also note that the the symbol for the scale parameter often is λ rather than β as used here.

Note that we here have assumed that even if we include the PM activity in the first turnaround, it will also be included in the second turnaround. In a more general setting, it might be that if we include now, we might skip the activity in the next turnaround. The analysis then becomes more complicated because to assess the costs that follow one needs to take into account different strategies after the second turnaround.

9.3 Single component - Impact on turnaround duration

Up to now we have considered that the cost of including the PM activity in the turnaround could be described by a single number, C_{TA}. Since the inclusion of any activity in a turnaround will influence the duration and complexity of the turnaround, it might be an oversimplification to consider a fixed cost incurred by the PM activity we are considering. At the end, we might still be able to calculate an expected cost for C_{TA}, but we will now investigate the situation in some detail.

Assume that there are several activities that are to be included in the turnaround, and that the PM activity we are considering is the *last* activity to include. For the time being, we assume that the duration of the turnaround is fixed, say D_{TA}. This is to say, that the planned duration is D_{TA}. The actual duration is a random quantity which we denote T_{TA}. Now, assume that we have an oversight over the remaining activities to include. The question is whether we should include the PM activity or not. Let $A_{\text{TA},0}$ denote the set of activities and their relations already considered to include in the turnaround. Further let $A_{\text{TA},1}$ be the activities to consider if we include the PM activity we are considering.

9.3.1 Consideration for inclusion in turnaround, $\tau^* > \tau_{\text{TA}}$ – No cancel possibilities

We now consider the same situation as in Section 9.2.3. The only difference is that C_{TA} is not known. We will assess the expected value of C_{TA}. To do this, we both consider the extra cost of conducting the PM activity (spare parts, man hour costs) etc. In addition we need to consider the influence wrt increasing the probability that we are not able to complete the turnaround in due time. Let $C_{\text{TA,F}}$ demote the fixed cost of the PM activity in terms of spare parts and man hour costs. Next let $C_{\text{TA,R}}$ denote the random cost related to the possibility of delaying the turnaround. The expected value of $C_{\text{TA,R}}$ may be found as the difference in expected production losses with and without the PM activity included. If the production loss per day is C_{PL} we have:

$$E(C_{TA,R}) = C_{\text{PL}} \int_{t=D_{\text{TA}}}^{\infty} [f_{T_I}(t) - f_{T_O}(t)]\,(t - D_{\text{TA}})\,dt \tag{9.16}$$

where $f_{T_I}(t)$ and $f_{T_O}(t)$ are the probability density functions for the project duration with and without the PM activity included respectively (I = Included, O = Outside the turnaround). D_{TA} is the due date for the turnaround, where there will be no production loss if project duration T is less than D_{TA}, and the production loss is T-D_{TA} if T is greater than D_{TA}. The situation is now identical to the situation in Section 9.2.3 if we set:

$$C_{\text{TA}} = C_{\text{TA,F}} + \text{E}(C_{\text{TA,R}}) \tag{9.17}$$

Thus the total cost of the PM activity execution within the turnaround is the fixed cost plus the random cost, $\text{E}(C_{\text{TA,R}})$, caused by the increased risk of exceeding the due date. Note that a first approximation is presented here. In reality the proses of select activities for inclusion is more challenging than considering one and one activity individually. Some kind of dynamic programming to select the most appropriate PM activities out of a huge list is required.

9.3.2 Consideration for inclusion in turnaround, $\tau^* > \tau_{\text{TA}}$ – Cancel possibilities

In Section 9.3.1 we considered a situation where the question was to determine whether to include the PM activity in the turnaround or not. If production losses are huge if the turnaround is delayed, we might in some situations choose another strategy where it is possible to cancel the PM activity if the turnaround for some reasons is delayed. Assume there is a critical milestone upfront of the execution of the PM activity. If a critical time is passed, one need to consider to skip the PM, and rather execute it at its optional time (τ^*). Let Let p_D denote the probability that the turnaround is delayed and one needs to consider to skip the PM activity as part of the turnaround. If the PM activity is skipped, it is assumed to be conducted at its optimal time, i.e., at τ^*-τ_{TA} time units after the turnaround. The decision diagram is shown in Figure 9.1.

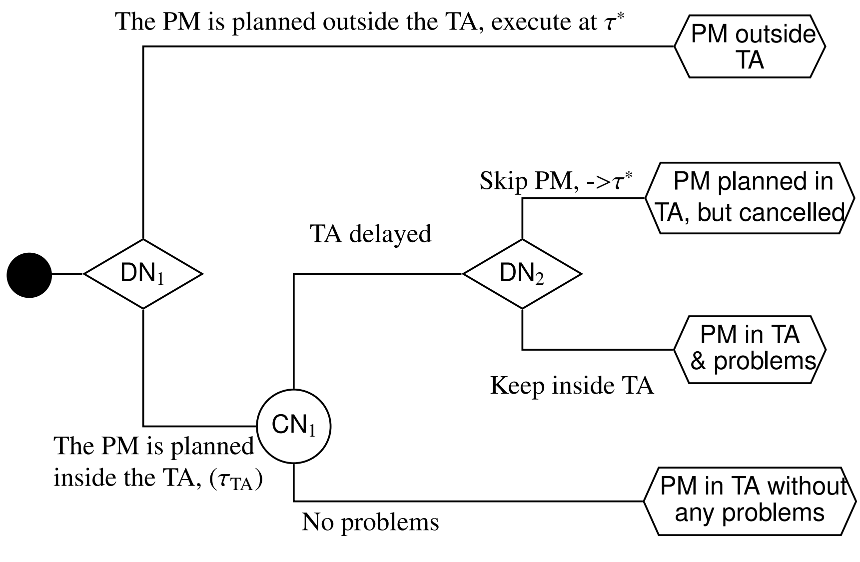


Figure 9.1: Decision tree, artic maintenance

The following quantities are now introduced:
$C_{\text{TA,P}}$: The preparation cost of executing the PM activity as part of the turnaround. This cost is to be paid independent of if the activity is executed as part of the turnaround or not.
$C_{\text{TA,E}}$: The execution cost of the PM activity if executed during the turnaround.
C_{PM}: Cost of executing the PM outside the turnaround, i.e., if executed at τ^*
p_D: the probability that the turnaround is delayed and one needs to consider to skip the PM activity as part of the turnaround
$f_{T_N}(t)$: The probability density function of the duration of the turnaround if no delays at the critical milestone
$f_{T_I}(t)$: The probability density function of the duration of the turnaround if delays at the critical milestone and the PM is kept within the turnaround.
$f_{T_O}(t)$: The probability density function of the duration of the turnaround if delays at the critical milestone and the PM is taken out of the turnaround.
C^*: Average cost per unit time if the PM is executed at its optimal time, i.e., at τ^*.
The remaining parameters required have been defined in previous sections.

In the calculation there are some challenges because the various end consequences in the decision three in Figure 9.1 has different time horizons. If the PM is executed within the turnaround, the next due time for PM is earlier than if the PM is executed at its optimal time, i.e., at τ^*. In order to have the same time horizon we assume that if the PM is executed during the coming turnaround it will also be maintained at the next turnaround. If the PM is not executed as part of the PM, but at its optimal interval at τ^*, the cost for the remaining time until the second turnaround is C^* per time unit. For the calculations we now assume that current time, t_0, correspond to the time when the coming turnaround is executed. The duration of the turnaround is so short that we may ignore this duration in comparison to τ_{TA}. Further we assume that the previous execution of the PM took place at time $t_0 - \tau_{\text{TA}}$, i.e., at the previous turnaround. The following cost equations may now be found for each end consequence, cf Figure 9.1:

PM outside TA

$$C_1 = C_{\text{PM}} + C_{\text{U}}\left[\lambda_{\text{E}}(\tau^*)\tau^* - \lambda_{\text{E}}(\tau_{\text{TA}})\tau_{\text{TA}}\right] + C^* \cdot (2\tau_{\text{TA}} - \tau^*) \tag{9.18}$$

PM planned in TA, but cancelled

$$C_2 = C_{\text{PM}} + C_{\text{TA,P}} + C_{\text{U}}\left[\lambda_{\text{E}}(\tau^*)\tau^* - \lambda_{\text{E}}(\tau_{\text{TA}})\tau_{\text{TA}}\right] + C^* \cdot (2\tau_{\text{TA}} - \tau^*) + C_{\text{PL}} \int_{t=D_{\text{TA}}}^{\infty} f_{T_O}(t)(t - D_{\text{TA}})dt \tag{9.19}$$

PM in TA & problems

$$C_3 = C_{\text{TA,E}} + C_{\text{TA,P}} + C_{\text{U}}\,\lambda_{\text{E}}(\tau_{\text{TA}})\tau_{\text{TA}} + C_{\text{PL}} \int_{t=D_{\text{TA}}}^{\infty} f_{T_I}(t)(t - D_{\text{TA}})dt \tag{9.20}$$

PM in TA without any problems

$$C_4 = C_{\mathrm{TA,E}} + C_{\mathrm{TA,P}} + C_{\mathrm{U}}\,\lambda_{\mathrm{E}}(\tau_{\mathrm{TA}})\tau_{\mathrm{TA}} + C_{\mathrm{PL}} \int_{t=D_{\mathrm{TA}}}^{\infty} f_{T_N}(t)(t - D_{\mathrm{TA}})dt \qquad (9.21)$$

With these cost figures the optimal decision is found by processing the decision tree in Figure 9.1.

9.3.3 "Arctic maintenance"

The situation to consider here is similar to the previous section, but we are now not allowed to execute the PM between two turnarounds. Also, if a failure occurs, we need to wait with the repair until the next opportunity, which here is set to the next turnaround. The situation is now illustrated in in Figure 9.2.

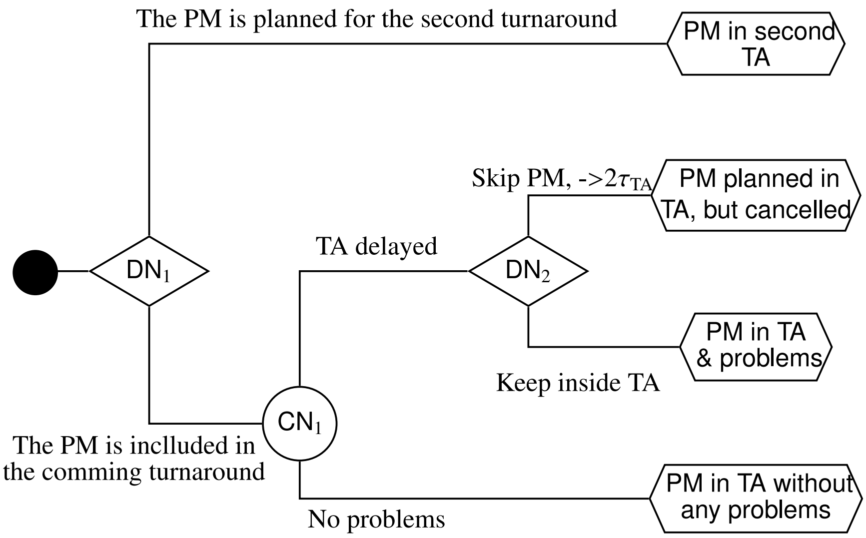


Figure 9.2: Decision tree, artic maintenance

9.4 Changing the frequency of the turnaround

Up to now the interval between turnarounds, τ_{TA}, has been considered fixed. In this Chapter we will establish models for extending the intervals. The following aspects will be taken into account:

- Currently there exist tasks that from a safety point of view need to be conducted at intervals of length $\tau_{\mathrm{TA},0}$, where $\tau_{\mathrm{TA},0}$ is the current turnaround interval. In order to extend the turnaround interval it is therefore necessary to improve the reliability of these components, or propose more extensively condition monitoring in the period between turnarounds. In the following we will only consider the possibilities to increase component reliability.

- Currently there exist tasks where the "optimum" interval is close to $\tau_{TA,0}$, and for these components we will also consider to increase the reliability by conducting upgrade projects prolonging the MTTF.

- $\tau_{TA,0} = 2$ years in the example, and the question to be raised is whether it pays off to extend the interval, i.e., $\tau_{TA} = 3$.

9.4.1 Treating only safety issues

One of the limitation with respect to extending the period between turnarounds is safety critical maintenance activities that need to be carried out at the current turnaround interval, $\tau_{TA,0}$. We now introduce the following quantities: {SC}: List of safety critical activity types where the current maximum interval is close to $\tau_{TA,0.}$
{NC}: List of activity types that are not safety critical.
n_i: Number of identical components representing activity type i.
$C_{UG,i}$: Cost of upgrading component i.
$C_{PM,i}$: Cost of executing preventive maintaining activity $i$outside the turnaround. This might be an option if it is too costly to upgrade.
$C_{TA,i}$: Cost of maintaining a component of maintenance activity type i as an integral part of the turnaround.
$C_{TA,B}$: Total cost of executing the turnaround treating only activities in the set {NC}, i.e., exclusive activities in {SC}.
MTTF$_i$: Mean time to failure for components with activity type $i \in$ {NC}.
α_i: Aging parameter for components with activity type $i \in$ {NC}.
$C_{U,i}$: Unavailability cost upon a failure for components with activity type $i \in$ {NC}.
$\lambda_{E,i} = \lambda_{E,i}(\tau_{TA}; \alpha_i, \text{MTTF}_i)$: effective failure rate of components with activity type $i \in$ {NC}.

Table 9.2: Reliability and cost data treating safety issues only

i	n_i	$C_{UG,i}$	$C_{PM,i}$	$C_{TA,i}$	MTTF$_i$	α_ι	$C_{U,i}$	$i \in$ {SC}
1	4	8	4	1				Yes
2	3	2	1	0.5				Yes
3	4	3	2	1				Yes
4	4			2	5	4	20	No
5	8			2	6	3	30	No
6	6			3	6	2.5	25	No

In addition we have for the example:$C_{TA,B} = 50$ (thousand NOKs).

We start by treating the average cost per year as a function of τ_{TA} assuming that the upgrading of safety critical components takes place:

$$C(\tau_{\text{TA}}) = C_{\tau\text{TA,B}}/\tau_{\text{TA}} + \sum_{i\in\{\text{SC}\}} n_i C_{\text{TA,i}}/\tau_{\text{TA}} + \sum_{i\in\{\text{NC}\}} n_i C_{\text{U,i}}\lambda_{\text{E},i}(\tau_{\text{TA}}) \tag{9.22}$$

where we have not taken discounting into account. By calculating $C(\tau_{\text{TA}})$ for $\tau_{\text{TA}} = 2$ and $\tau_{\text{TA}} = 3$ in equation (22) we may calculate the yearly gain or loss by changing the interval of the turnaround. If there is a gain, we also need to consider the (yearly) cost of the upgrading project:

$$C_{\text{UG}} = \sum_{i\in\{\text{SC}\}} n_i C_{\text{UG},i}/T \tag{9.23}$$

where T is the number of years to consider, e.g., $T = 10$ if the installation will be disposed after 10 years. C_{UG} should not exceed the gain by increasing τ_{TA} from 2 to 3 years.

Problem 9.1 Consider the example data above, and calculate the yearly cost for $\tau_{\text{TA}} = 2$ and $\tau_{\text{TA}} = 3$ in order to determine if the turnaround interval could be increased. □

It is a simplification to ignore the discounting of the cost elements in this situation. We will now discuss how to discount the variable cost related to failures. $\lambda_{\text{E}}(\tau_{\text{TA}})$ represents the yearly expected number of failures if the time unit is year as we assume in this example. It is then reasonable to calculate the expected number of failures each year j, $j = 1,..,\tau_{\text{TA}}$ and discount the cost to present values. We assume that all costs are incurred at the end of the year, although it could also been argued that it is more reasonable to assume that a failure occur in the middle of a year. We now introduce $\Lambda_{E,i}$ as a measure of "discounted number of failures" in a period between two turnarounds where the turnaround interval is τ_{TA}:

$$\Lambda_{\text{E},i}(\tau_{\text{TA}}, r) = \sum_{j=1}^{\tau_{\text{TA}}} \left(j\lambda_{\text{E},i}(j) - (j-1)\lambda_{\text{E},i}(j-1)\right)(1+r)^{-j} \tag{9.24}$$

where r is the interest rate. Further we let $n(T,\tau_{\text{TA}})$ be the number of turnarounds for an installation that is disposed in year T (no turnaround the last year) when the turnaround interval is τ_{TA}. Table 9.3 shows $n(T, \tau_{\text{TA}})$ as a function of τ_{TA} and T.

Table 9.3: $n = n(T, \tau_{\text{TA}})$ for various combinations of τ_{TA} and T

$\tau_{\text{TA}}\backslash T$	8	10	12	15	20
1	8	10	12	15	20
2	4	5	6	8	10
3	3	4	4	5	7
4	2	3	3	4	5

We always assume that the turnaround is executed at time $t = 0$, then after time τ_{TA}, $2\tau_{TA}$ and so on. We still assume that the upgrade of all safety critical components takes place. The turnaround related cost up to time of disposal is given by:

$$C^A(\tau_{TA}) = \sum_{j=1}^{n(T,\tau_{TA})} (1+r)^{-(j-1)\tau_{TA}} \left(C_{TA,B} + \sum_{i\in\{SC\}} n_i C_{TA,i} + \sum_{i\in\{NC\}} n_i C_{U,i} \Lambda_{E,i}(\tau_{TA}, r) \right) + \Delta C_{FLP} + \sum_{i\in\{SC\}} n_i C_{UG,i} \quad (9.25)$$

where ΔC_{FLP} is a correction term to account for failures the last period, i.e., the difference between $\Lambda_{E,i}(\tau_{TA},r)$ calculated after the last turnaround, and the actual discounted number of failures multiplied with the failure cost and summed over all activities that are not safety critical. There might be a difference because either we skip a turnaround the last year and hence there is an extra failure cost for that year not accounted for, or because we accumulate failure costs in the calculation formula for the period after T. If T is large, i.e., in the order of magnitude 10 years, we may ignore ΔC_{FLP} due to discounting effects.

The accumulated turnaround related cost for the remaining life of the installation may now be compared by applying equation (9.25) for $\tau_{TA} = 2$ and $\tau_{TA} = 3$ respectively, where we for $\tau_{TA} = 2$ do not include upgrade costs, $C_{UG,i}$.

If upgrade cost, $C_{UG,i}$, is high for some components we may also consider to exclude the corresponding activities as part of the turnaround. This means to remove the corresponding terms for $n_i C_{TA,i}$ and $n_i C_{UG,i}$ in equation (refeq:turnaroundLCC1), and similarly add the cost of preventive maintenance: $\sum_{j=1}^{n(T,\tau_{TA,0})} (1+r)^{-(j-1)\tau_{TA,0}} n_i C_{PM,i}$ to the total cost for τ_{TA} equal the extended turnaround interval (i.e., $\tau_{TA} = 3$) and $\tau_{TA,0}$ is the original turnaround interval.

Problem 9.2 Consider the example data above, and calculate the total accumulated cost for the remaining period up to year T for $\tau_{TA} = 2$ and $\tau_{TA} = 3$ in order to determine if the turnaround interval could be increased from a life cycle cost perspective. □

Note that the cost terms $C_{UG,i}$ might contain some implicit interrelations. Since the upgrade is part of the next turnaround, it is reasonable that adding more and more upgrade projects totally increases the cost more than the individual activities requires due to the increase of complexity, and impact on the shutdown duration. This is not pursued further in this presentation.

9.4.2 Treating production related issues

For production related activities we also need to consider whether an upgrade of the components could have an impact on the reliability. We here assume that if we upgrade a component we could increase the corresponding MTTF. In the following we assume that the upgrade cost per production related component is $C_{UG,i} = 2$, which will increase MTTF with 25%. In the optimization we therefore need to consider the following options:

1. Keep the activity in the turnaround, do not upgrade
2. Keep the activity in the turnaround, upgrade in order to prolong the MTTF
3. Take the activity out of the turnaround, and pay a higher cost $C_{\mathrm{PM},i}$ but with the flexibility this gives to apply the optimal individual maintenance interval. We assume that $C_{\mathrm{PM},i} = 1.5C_{\mathrm{TA},i}$.

If there are no interrelations between upgrading cost of the production related activities we may perform individual analysis for each of the activity types. If there are interrelations the situation becomes much more complicated. A dynamic programming approach might be required, or at least we need to consider the activities pair wise.

In the following we assume that we may treat each activity individually.

Keep the activity in the turnaround, do not upgrade

In this situation we apply equation (9.25) as it is written. By extending the turnaround interval there will be fewer turnarounds to be executed, but the term $\Lambda_{\mathrm{E},i}(\tau_{\mathrm{TA}}, r)$ will become more significant since the effective failure rate increases at the end of the period between turnarounds.

Keep the activity in the turnaround, upgrade in order to prolong the MTTF

We apply equation (9.25) but need to add the cost of upgrading, i.e.,$n_i C_{\mathrm{UG},i}$. Since MTTF is prolonged the term $\Lambda_{\mathrm{E},i}(\tau_{\mathrm{TA}}, r)$ will also be reduced.

Exclude the activity from the turnaround

We apply equation (9.25) but need to reduce the term $C_{\mathrm{TA,B}}$. As a first approximation we subtract the cost of executing the preventive maintenance activity as part of the turnaround, i.e., $C_{\mathrm{TA,B}} = C_{\mathrm{TA,B}} - n_i C_{\mathrm{TA},i}$. We also need to remove the corresponding failure cost terms in equation (9.25). But since we have to execute the PM activity i outside the turnaround we now add:

$$\sum_{j=1}^{n(T,\tau_i^*)} (1+r)^{-(j-1)\tau_i^*} \left(n_i C_{\mathrm{PM},i} + n_i C_{\mathrm{U},i} \Lambda_{\mathrm{E},i}(\tau_i^*, r)\right) \tag{9.26}$$

Note that τ_i^* is not an integer so we may need to rewrite the $n()$ and $\Lambda()$ functions.

Problem 9.3 Assume that we increaseτ_{TA} to 3 years. For each $i \in \{\mathrm{NC}\}$find out whether it pays off to pay the upgrading cost. □

9.5 Risk identification for critical activities

In this section we will consider critical activities in the turnaround. We will only treat those activities that relate to an upgrade of the flare which is a main part of the turnaround. The activities are listed in Table 9.4:

The relation between the activities is as follows:

Table 9.4: Activities to include and durations (normal conditions)

ID	Activity	L	M	H
0	Shutdown	1	2	3
A	Flare gas meters	1	2	3
B	Change flare tip	1	2	3
C	LP Flare drum	4	5	6
D	NF HP Flare drum	3	4	5
E	HP Flare drum	3	4	5
F	Install new Flare tip system	5	6	8

- 0 – Shutdown is conducted first, then follows in parallel:
  - A - Flare gas meters/B - Change flare tip (following each other)
  - C - LP Flare drum
  - D - NF HP Flare drum
  - E - HP Flare drum
- Activity: F - Install new Flare tip system, may start when all the other activities are completed.

In Table 9.5 we have listed some threats related to activity: B - Change flare tip. We will consider only those activities where a schedule impact is explicitly listed. The turnaround is scheduled for 14 days. It cost 10 million NOKs each day the turnaround is delayed. Costs in Table 9.5 are given in million NOKs.

Problem 9.4 Conduct a cost benefit analysis of the measure where one wants a more powerful helicopter. □

Table 9.5: Threats related to activity: B - Change flare tip

Threat element	Probability	Schedule impact	Comment/Measure	Cost
Technical problem with helicopter or equipment	0.05	1	Technician on type (TLM) be available for repair.	0.05
Failure in clearance of lifting	0.01		Accepted and tested strap arrangement for the flare tip, specified by OM.	
Wind force over 30 knots	0.08	1 per windy day	Given wind force over 30 knots (critical limit) the day of execution, the probability of the wind problem repeats the subsequent days is 50% for each day. Proposed measure: More powerful helicopter that may operate in stronger wind. To model the impact, we assume that all the probabilities are reduced by 30%	1
Failure in radio communication	0.02		Stop all operations until the problem is solved/re-established.	
Loosing object fly around rotor wind	0.01		Insist take off and delivering area. Cancel the take-off and postpone operation until the area is safe. This has to be a continuous process.	
Failure with lifting strap	0.03		Use of certificated and approved lifting equipment. Prohibited zone for flying with hanging load has to be defined.	
Failure with lifting hook	0.02		All equipment are to be function test before operation. Prohibited zone for flying with hanging load has to be defined.	
Unexpected rotation on load	0.05		Use of certificated and approved straps and lifting equipment. In case the pilot can't stop rotation by changing of the speed, the load has to be return back to helideck and defined again.	
Failure with motor or loss of motor craft	0.01		Personnel shall not stay under the loading zone where the load is lowering.	
Damage of flare tip	0.05	5	It will take 10 extra days to require a new flare tip if it is damaged during the lifting operation. A spare flare tip might be stored onshore, and requested in case of damage. It takes one extra day to bring it offshore	0.5

Chapter 10

Portfolio management

10.1 Basic probability notation

10.2 Introduction

A project portfolio is a group of projects to be carried out under the sponsorship of a particular organization. The portfolio may be seen from various stakeholders. For example it might be a set of oil fields to be developed for a major oil company. Further, it might be a set of installation project for a subsea equipment manufacturer. A third stakeholder will be a financial institute (banks etc) supporting the project financially in a limited time period. In this memo we will discuss the following:

- Visualization of project portfolios
- Selection of projects for execution
- Vulnerability of project portfolios
- Scheduling of projects within a portfolio

10.3 Visualization of project portfolios

There are various ways to visualize the projects within a potential portfolio. Such a visualization is valuable to get an oversight of a huge number of projects and get a preliminary oversight of these with respect to which ones to go for. We will now, and in the following assume that for the individual projects key numbers or characteristics have been derived. Examples of such numbers are:

- Expected net present value (NPV) of the project
- Standard deviation of the NPV
- Probability that NPV <0

- Probability that actual cost is larger than a factor, f, times the budget asked for, for example f =1.5, and f = 2
- Probability of major failure of technology development
- Return factor, i.e., return (in Euro) per Euro invested

Bubble diagrams are convenient to visualise tree of the dimensions at a time. To cover more dimensions several bubble diagrams are required. We will consider the data shown in Table 10.1:

Table 10.1: Project key numbers

Project Name	NPV	Pr(Cost overrun)	Return ratio
A	10	0.3	3
B	5	0.2	2
C	8	0.1	2
D	4	0.45	5
E	7	0.3	4
F	4	0.2	3

The net present value (NPV) represents the total discounted cost and income in the project, either for a period of years, or an infinite time horizon if relevant. The corresponding bubble diagram is shown in Figure 10.1.

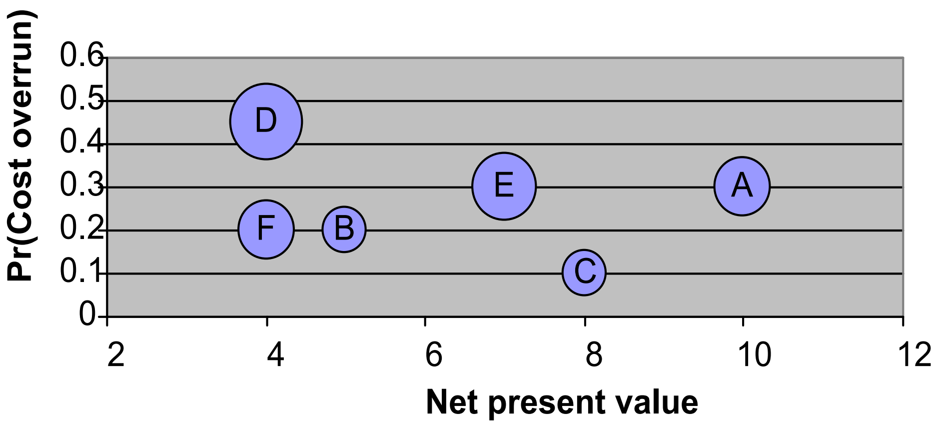


Figure 10.1: Example of bubble diagram where the size of the bubbles represents the return ratio

10.4 Selection of projects for execution

A first approach to select projects will be to combine the key numbers describing each project. Now, let x_{ij} denote key number j for project i. Further, assume that each variable has been given a weight, w_j. In order to make the presentation as simple as possible we assume that the weights are positive if a high value of x_{ij} is preferred, and the weights are negative if a low value of x_{ij} is preferred. A score for project i may be achieved by

$$S_i = \Sigma_j w_j \cdot x_{ij} \quad (10.1)$$

We return to the example presented in Table 10.1, where the following weights has been introduced:

w_1 = 4 = Weight of (expected) NPV

w_2 = -2 = Weight of probability of overrun (e.g., more than a factor of 1.5)

w_3 = 3 = Weight of return ratio

The calculated costs are shown in Table 10.2. The most promising projects according to the applied weights are A, E and C respectively. Projects might be chosen according to the calculated scores, where budget constraints might limit the number of projects. Assume that the budget is **8 units**.

Table 10.2: Example data with calculated scores

Project Name	NPV	Pr(Cost overrun)	Return ratio	Project cost	Score
A	10	0.3	3	3.3	48.4
B	5	0.2	2	2.5	25.6
C	8	0.1	2	4.0	37.8
D	4	0.45	5	0.8	30.1
E	7	0.3	4	1.8	39.4
F	4	0.2	3	1.3	24.6

Starting from the top of the list sorted on the score gives projects A, D, E and F where we always fulfil the budget constraints. We then spend 7.2 cost units, and the total score of the selected projects is 142.5. In general the problem is a dynamic programming problem. In order to set the weights often the Analytic Hierarchy Process (AHP) is applied, see e.g., [31].

Note that the score of each project is a weighted sum of various criteria. It is, however, not obvious that such a weighted score make sense in selecting a *range of project*. If there are budget constraints as indicated above, the best strategy would obvious be to chose project with a low cost and a reasonable high score. An alternative approach would be to derive a "utility" based score. Ideally we could find a utility function and then search for a portfolio of projects that maximizes expected utility under the budget constraints. By introducing a utility function the probability of cost overrun could be discarded from the utility function since the uncertainty related to overrun is reflected in the concave utility function. The return ration would also be of less importance. Obvious it seems reasonable to consider the return ration, but it might be argued that how much we get back of the money we invest is not the most appropriate measure, the most interesting measure is the total profit at the end of the day. A pragmatic approach is now to use the NPV value as the baseline for the score of a project, but introduce a penalty for project with high probability of cost overrun, and projects with high cost, i.e., those projects that lock capital. A proposed measure is:

$$S_i^* = \text{NPV}\,(1 - 0.3\,\text{Pr}\,(\text{CostOverrun})) - 0.2 \cdot \text{ProjectCost} \quad (10.2)$$

where the weights 0.3 and 0.2 represent how negative overruns and lack of capital are considered respectively.

Table 10.3: Project summary, alternative score S*

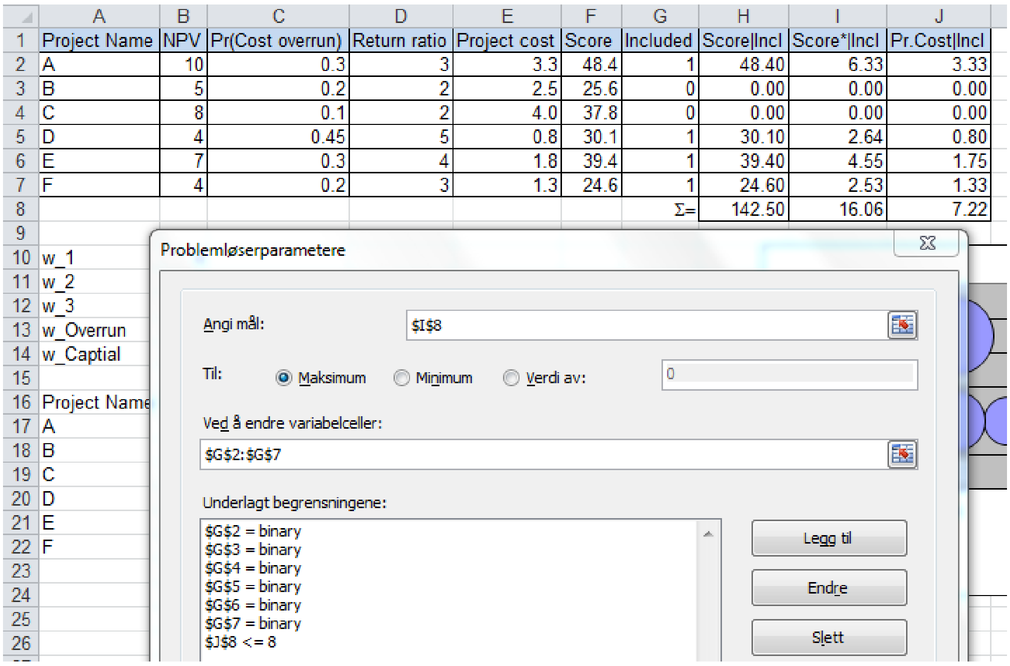
Project Name	NPV	Pr(Cost overrun)	Return ratio	Project cost	Score	Included	Score |Incl	Score* |Incl	Pr.Cost |Incl
A	10	0.3	3	3.3	48.4	1	48.40	6.33	3.33
B	5	0.2	2	2.5	25.6	0	0.00	0.00	0.00
C	8	0.1	2	4.0	37.8	0	0.00	0.00	0.00
D	4	0.45	5	0.8	30.1	1	30.10	2.64	0.80
E	7	0.3	4	1.8	39.4	1	39.40	4.55	1.75
F	4	0.2	3	1.3	24.6	1	24.60	2.53	1.33
						Σ =	142.50	16.06	7.22

Table 10.3 is prepared for using the Excel solver to optimize which projects to include. In the column "Included" a binary variable is introduced, where 1 means that the project is included, and 0 means that the project is not included. In the Excel solver all the "Included" variables are defined as binary variables in the "Constraints" (Underlagt begrensningene) in Figure 10.2. Further the sum of the project cost (cell J8) is limited to 8 units. The target cell is here set to cell I8 for the alternative score, S*. The optimization in this particular situation gives the same result as for the original score. The reason for this is that project C locks much capital even though the score is high (S* = 4.8 which is the second highest) .

10.5 Vulnerability of project portfolios

The figures in Table 10.2 represent expected values. These numbers have most likely been assessed by various risk analyses, hence we often have uncertainty distributions over the key figures. In portfolio analysis we distinguish between random variation and systematic variation. Random variation represents variation within one project independent of the other projects. Systematic variation represents variation due to one or more factor possible affecting two or more projects. In portfolio analysis it is the systematic variation which usually is of concern. The random variation will be wiped out due to the "law of large numbers". We will therefore in the following discuss systematic variation. We now propose a 8 step method for selecting projects taking systematic variation into account.

1. Calculate key numbers for relevant projects as described in Section 10.4



	A	B	C	D	E	F	G	H	I	J
1	Project Name	NPV	Pr(Cost overrun)	Return ratio	Project cost	Score	Included	Score|Incl	Score*|Incl	Pr.Cost|Incl
2	A	10	0.3	3	3.3	48.4	1	48.40	6.33	3.33
3	B	5	0.2	2	2.5	25.6	0	0.00	0.00	0.00
4	C	8	0.1	2	4.0	37.8	0	0.00	0.00	0.00
5	D	4	0.45	5	0.8	30.1	1	30.10	2.64	0.80
6	E	7	0.3	4	1.8	39.4	1	39.40	4.55	1.75
7	F	4	0.2	3	1.3	24.6	1	24.60	2.53	1.33
8							Σ=	142.50	16.06	7.22
9										
10	w_1									
11	w_2									
12	w_3									
13	w_Overrun									
14	w_Captial									
15										
16	Project Name									
17	A									
18	B									
19	C									
20	D									
21	E									
22	F									
23										
24										
25										
26										

Figure 10.2: Setup for optimization in Excel

2. Choose the most promising projects based on a preliminary score (S_i), but include more projects than the total budget constraints

3. Identify one or more vulnerability factor affecting two or more projects

4. For each project the vulnerability factors are considered explicitly. Further for each combination of the vulnerability factors recalculate the expected values of the key numbers. Let $\{V\}$be the set of all combination of the vulnerability factors.

5. Present the corresponding projects key figures by bubble diagrams for the corresponding values of $\{V\}$in order to visualise the impact of the vulnerabilities.

6. Recalculate the expected values of the weighted sum for various subsets of projects using the law of total probability, where p_v is the probability for element v in $\{V\}$. This means that we for each v in $\{V\}$first calculate the sum of scores, $\Sigma_i S_i\,|v$, for projects included by conditioning on the value v, then we sum these weighted scores over all v in $\{V\}$in a new weighted sum using the p_v-values.

7. Calculate also the variance of the weighted sum.

8. Make a final selection of projects based on the vulnerability calculations.

9. Highlight the vulnerability factors as an important input to the overall portfolio management (e.g., risk registers for each project, and global enterprise risk registers).

Table 10.4 shows the situation where only one vulnerability factor is considered. Two values are considered, ☺ = Good, ☹ = Bad.

Table 10.4: Key project figures with vulnerability included

Project Name	NPV ☺	NPV ☹	Project cost ☺	Project cost ☹	Return ratio ☺	Return ratio ☹
A	11	3	3	7	3.7	0.4
B	6	4	2.5	4	2.4	1.0
C	8.5	2	3	7	2.8	0.3
D	4.5	3	0.6	1.5	7.5	2.0
E	8	4	1.5	4	5.3	1.0
F	4.5	3	1.2	2	3.8	1.5

We assume that Pr(☺) = 0.8, and Pr(☹) = 0.2.

Problem 10.1 Discuss steps 1-8. Maximise expected NPV under the constraints that the total spending given the bad value (☹) of the vulnerability factor should not exceed 12. □

10.6 Scheduling of projects within a portfolio

Given a selected portfolio of projects we need to schedule them in time. In some situations the scheduling might be unproblematic, but in other situations we need to take resource constraints into account. We will in the following only consider one critical resource (e.g., labour force of a critical discipline). We assume that we have done a preliminary assessment of resources required in each project in the various project phases. We first assume that both resources and durations are deterministic quantities. We will also assume that the order the projects are started are determined by other factors than resources considerations. Table 10.5 shows the example data we will use. We assume that all projects run through three phases, where the resource demand is fixed in each phase. The D-values shows the durations of each phase given the required resources, i.e., the R-values. Total available resources for the critical discipline are **4**.

It is reasonable to assume that if one activity is run with less resources than required the duration will be longer than given in Table 10.5. We assume that if an activity in one of the phases is run with available resources $R < R_0$ = Required resources, then the actual duration of this phase will be:

$$D = D_0 \cdot f \cdot R_0/R \tag{10.3}$$

Table 10.5: Resources required for each project during various project phases

Project	Start	$D1_0$	$R1_0$	$D2_0$	$R2_0$	$D3_0$	$R3_0$
Project 1	0	3	2	5	3	3	1
Project 2	5	2	1	4	2	1	1
Project 3	7	5	2	3	3	2	1

where D_0 is the duration under the original setup, f is a factor accounting for complication due to more difficult coordination with other resources, the planning challenges etc. In a detailed model we might be able to be more explicit on the modelling, and need not to include such a "correction" factor. However, usually such detailed models will not be available when projects are seen together as part of portfolio management. We will assume that $\boldsymbol{f} = \mathbf{1.25}$. The available recourses for all projects may vary as a function of time, but in the following we simplify and assume that the maximum available recourses to share for all activities are R_{Max}. The activities have to fight about the available resources. To find the optimal allocation of resources for each project is not a simple task. It will not be possible to present a formula for the optimal allocation. To solve such an allocation problem we may simulate the progress in each project. The following issues are included in the model:

1. The start-up of each activity is a decision variable, i.e., a vector of project start-up points.
2. An evaluation point is a point of time where one project enters a new phase, or is finalized
3. At any evaluation point resources for corresponding activities entering a new phase are first released to the pool of resources, then available resources will be allocated to the project that has the longest expected time to finalization.
4. To find expected time to finalization, we first calculate progression in current activity, then we assume that from now on the project achieve full resource allocation for the remaining part of the current activity, and the subsequent phases.
5. If there still are recourses left, they are allocated to the project with the second longest expected time to finalization.

Problem 10.2 Find the optimal starting point for project 2 and project 3 when you assume that when each project is completed the income is 1 unit per time unit. Further we assume that the cost of execution of each project is 0.1 per time unit the project is run. I.e., an early "soft" start is not "free of charge". Total available resources for the critical discipline are $R_{\text{Max}} = 4$.

□

Problem 10.3 Repeat the problem above, but now assume that the durations are PERT distributed with most likely value equal to the values in Table 10.5, and the low and high values deviate 20% from the most likely value □

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